STOCHASTIC STEFAN PROBLEMS: EXISTENCE, UNIQUENESS, AND MODELING OF MARKET LIMIT ORDERS

BY

ZHI ZHENG

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

Doctoral Committee:

Assistant Professor Robert Edward DeVille, Chair
Associate Professor Vadim Zharnitsky, Contingent Chair
Professor Richard B Sowers, Director of Research
Assistant Professor Zoi Rapti

Abstract

In this thesis we study the effect of stochastic perturbations on moving boundary value PDE's with Stefan boundary conditions, or Stefan problems, and show the existence and uniqueness of the solutions to a number of stochastic equations of this kind. We also derive the space and time regularities of the solutions and the associated boundaries via Kolmogorov's Continuity Theorem in a defined normed space.

Moreover, we model the evolution of market limit orders in completely continuous settings using such equations, derive parameter estimation schemes using maximum likelihood and least mean-square-errors methods under certain criteria, and settle the investment optimization problem in both static and dynamic sense when taking the model as exogenous.

To My Beloved Parents and Family.

Acknowledgments

This thesis would not have been possible without the support of many people.

First I wish to thank my thesis advisor, Prof. Richard B. Sowers, for the unlimited support and guidance he has provided with me, which includes not only the technical discussions on my research problems, but more importantly, many of the inspirations and intuitions about the research directions and the methodologies in conducting research as well. I have also improved my academic writing by learning from his work.

I also thank Prof. Robert E. DeVille, Prof. Zoi Rapti, and Prof. Vadim Zharnitsky for serving on my thesis committee. I am also grateful to Prof. Kay Kirkpatrick, Prof. John Marden, Prof. Nolan Miller, and Prof. Renming Song for their kind assistance and advice during my doctorate research.

There are many other people I also feel grateful to. As a student who spent the first year in Department of Statistics and the next four years in Department of Mathematics, I have made many friends in those two departments, and those five years with them have been one of the most precious periods of time within my past life.

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Chapter 1

Introduction

An important type of problems in the theory of partial differential equations (PDE's) is the moving boundary value problems. In this thesis we study the effect of stochastic perturbations on such type of problems with Stefan boundary conditions, or Stefan problems, which have various applications in physics, engineering, and finance, and show the existence and uniqueness of the solutions to a number of stochastic equations of such kind. We also obtain the space and time regularities of the solutions and their associated boundaries via Kolmogorov's Continuity Theorem in a defined normed space.

Moreover, based on the nature of the stochastic version of the problem, we model the evolution of market limit orders in completely continuous settings using such equations, derive parameter estimation schemes using maximum likelihood and least mean-square-errors methods under certain criteria, and settle the investment optimization problem in both static and dynamic sense when taking the model as exogenous.

1.1 Mathematical Background

A moving boundary PDE of $u(t, x)$ describes the behavior of a system that consists of two phases, as illustrated in Figure 1.1, where $\beta(t)$ is a moving boundary which is part of the solution and must be solved

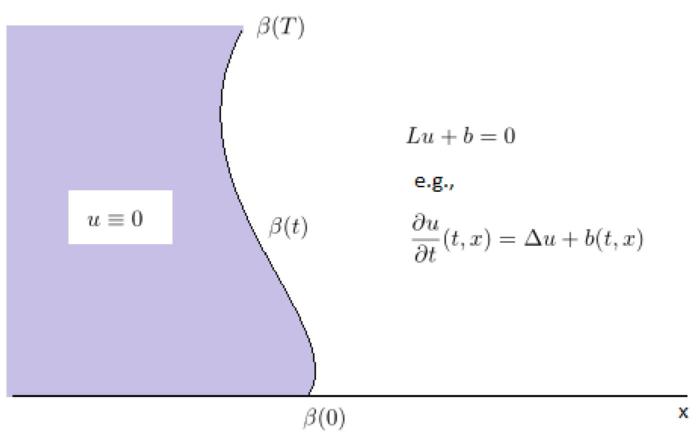

Figure 1.1: An Illustration of Moving Boundary PDE's

simultaneously with $u(t,x)$. As can be seen in Figure 1.1, in the region to the left of the moving boundary (namely, the set $\{(t,x) \in [0,\infty) \times \mathbb{R} : x \leq \beta(t)\}$) u is constantly set to 0; on the right side of the boundary (the set $\{(t,x) \in [0,\infty) \times \mathbb{R} : x > \beta(t)\}$) u is described by a PDE of the general form $Lu + b = 0$ where L is a predefined second-order differential operator.

For a moving boundary PDE problem, in addition to the regular boundary condition such as the Dirichlet condition $u(t,\beta(t)) = 0$, there is always an extra boundary condition that describes the dynamics at the moving boundary, for instance, the *Stefan boundary condition*

$$\frac{\partial u}{\partial x}(t,\beta(t)+) = \rho\dot{\beta}(t). \tag{1.1}$$

The type of moving boundary PDE's we consider throughout the thesis is the *Stefan problems*, where L is a heat or parabolic operator (for instance, $L := -\partial/\partial t + \partial^2/\partial x^2$) with the Stefan boundary condition. Such type of problems has a variety of applications. For instance, in physics, they model the phenomena such as ice melting with the Stefan condition describing the heat balance at the interface (the moving boundary, see [3]); in finance they model the valuation of American options with the PDE derived from the Black-Scholes formula and the moving boundary describing the early exercise price boundary (see Lemma 7.8, Chapter 2, [5]).

1.2 Motivation and Mathematical Results

In the mathematics of this thesis we are interested in the stochastic versions of the Stefan problems, namely, $b(t,x)$ is a formal notation about the stochastic addition (noise). In general $b(t,x)$ is a 2-dimensional distribution and therefore we work within the framework of the stochastic PDE theory by Walsh in [10], and based on the weak formulations of the equations and their equivalent evolution equations.

When $b(t,x)$ is multiplicative, namely, $b(t,x) = u(t,x)\dot{W}(t,x)$ where $\dot{W}$ is the noise (formal), [6] proves the existence and uniqueness of the solution when $\dot{W} = \dot{W}(t)$ is a distribution (Brownian) only in time and is constant in space. Also, in [7] we proved the existence and uniqueness of the solution when $\dot{W}(t,x)$ is a distribution (Brownian) only in time and is smoothly correlated ("colored") in space, which is quickly reviewed in Chapter 2.

However, when $\dot{W}(t,x)$ is a distribution (Brownian) in both space and time, we face a number of novel challenges that are beyond the scope of current literature:

(1) The spatial derivatives of the solution may not exist (see [10]), which means the techniques we used in [7] based on H-norms may not be available; more importantly, the Stefan condition (1.1) involves

the spatial derivative of the solution at the boundary, and therefore we shall show the existence of such a derivative simultaneously with the existence and uniqueness of the solution.

(2) The stochastic perturbation given by $b(t,x)$ is no longer spatially Lipschitz as in [6] and [7], which means in order to control the boundary shift effect in the Itô integrals and the nonlinear drift term we may need additional spatial regularities on $b(t,x)$ in addition to just being multiplicative as in [6] and [7]; in other words, although the perturbation b vanishes when $u=0$, u itself may not provide sufficient spatial smoothness to control the Itô integrals when the boundary shifts.

To tackle (1) alone, in Chapter 3 we first study the stochastic heat equation driven by a multiplicative space-time Brownian noise, namely, with W a standard 2-dimensional Brownian sheet defined in [10],

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial^2 W}{\partial t \partial x}, \forall x > 0, t \in [0,T], \\
u(0,x) &= u_0(x), \\
u(t,0) &= 0, \forall x \leq 0, t \in [0,T].
\end{aligned}$$

From such study we obtain the conditions and form of norms under which a boundary derivative exists, and more importantly, develop the essential techniques to calculate the sample-wise space and time modulus of continuity of the solution by means of Kolmogorov's Continuity Theorem (see [4]), which is critical to evaluate the regularity needed to control the boundary shift effect.

Next, in Chapter 4, first with the same multiplicative space-time Brownian noise (that is, $b(t,x) = uW_{tx}$,) we preliminarily calculate the effect of a boundary shift on the Itô integral using the techniques and results obtained in the previous chapter, and find that it may be difficult to obtain the wanted control on iteration. Therefore we make the change in the stochastic perturbation $b(t,x)$ such that it is even smoother in space and in turn study the following stochastic Stefan problem driven by a scaled space-time Brownian noise:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + \sigma(x-\beta(t))\frac{\partial^2 W}{\partial t \partial x}, \forall x > \beta(t), \\
u(t,x) &= 0, \forall x \leq \beta(t), \\
u(0,x) &= u_0(x), \\
\rho\dot{\beta}(t) &= \lim_{x \searrow \beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned}$$

where $b(t,x) = \sigma(x-\beta(t))W_{tx}$ and $\sigma(\cdot)$ is a function that satisfies certain regularity conditions which are sufficient to tackle (2). We show the existence and uniqueness of the solution to the above equation and additionally the regularity of the boundary using results from this and previous chapters. This result also serves as the mathematical foundation for the modeling of the dynamics of limit orders.

1.3 Modeling of Market Limit Orders

As an application of the mathematical results, in Chapter 5 we model the evolution of market limit orders using stochastic Stefan equations with a number of model parameters. In literature there have been researches on the modeling of market limit orders and their execution such as [8] and [9], most of which are based on discrete settings, for instance, models based on Poisson processes and/or queuing theory. However, because of rapid technological evolution which brings about ultra-fast microprocessors and hardware, trading behaviors and patterns involved with a large amount of limit order creation, transaction, and cancellation within a short period of time, such as high frequency trading (HFT), have become quite popular and tend to have a heavy impact on the mechanisms of price discovery and formulation. Consequently, a continuous and dynamic model of the evolution of limit orders in a particular market and their dynamics is proposed, which is described by stochastic Stefan equations with a number of model parameters to be estimated given a real dataset.

In Section 5.1 we model the evolution of limit orders in a particular market by Stefan equations according to the following facts, and show the existence of such model based on the mathematical results obtained in the previous chapters.

(1) Limit orders are placed, cancelled, and executed in a manner where jitters tend to be rapidly smoothed out, which is why we have a Laplacian;

(2) The change of the mid-price is driven by the intensity of interaction between ask and bid orders around the mid-price;

(3) The randomness comes from the constant creation, cancellation, and execution of limit orders; its intensity varies at different (limit) prices, and tends to vanish as the price goes far beyond the mid.

In Section 5.2 we study the methods to estimate the model parameters based on a given limit order dataset and derive the statistics such as a Maximum-Likelihood Estimator (MLE) and an estimator that minimizes the Mean-Square Errors (MSE), both under AIC (see [1]) since the number of parameters (or dimension) is to be estimated itself. Under certain simplified (or degenerated) circumstances an explicit expression of an MLE estimator is also derived.

In Section 5.3 we study the model and method to maximize the utility function of an investor who takes the model as exogenous and find the optimal limit-price-to-buy (or equivalently, amount-to-buy) of the asset and the consumption, given a fixed amount of wealth at a given time. Both static and dynamic analysis are studied to obtain the optimality of the investment via the limit order model, and two theorems are given respectively for those two types of analysis as the criteria to test for optimality, which also have intuitive interpretations.

Chapter 2

A Stochastic Stefan Problem with Spatially Colored Noise

In this chapter we quickly review our work in [7] by giving the main theorems and lemmas without proofs. We studied a stochastic Stefan problem of $u(t,x)$ driven by a multiplicative noise $u(t,x)d\xi_t(x)$, where $\xi_t(x)$ is a noise that is Brownian in time and smoothly correlated (or "colored") in space. Specifically, fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose $W : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the standard 2-dimensional Brownian sheet. Suppose also that $\eta : \mathbb{R} \to \mathbb{R}$ is C^∞ and $\|\eta\|_{L^2(\mathbb{R})} = 1$. Then for $t \geq 0, x \in \mathbb{R}$, define

$$\xi_t(x) := \int_0^t \int_0^\infty \eta(x-y) W(dyds).$$

Then we have the following problem and results.

2.1 Problem and Main Theorem

We showed the existence and uniqueness of the solution $u(t,x)$ to the following formal equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + u(t,x)d\xi_t(x), \forall x > \beta(t), \\
u(t,x) &= 0, \forall x \leq \beta(t), \\
u(0,x) &= u_0(x), \\
\rho\dot{\beta}(t) &= \lim_{x \searrow \beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{2.1}$$

for $0 \leq t < \tau$ where τ is some well defined stopping time.

Since W is a distribution, equation (2.1) is in fact formal, and we need to work on the weak definition (Definition 3.2 in [7]) and its equivalent evolution equation (Equation (15) or Lemma 3.4 in [7]). The main theorem is

Theorem 2.1.1 *The solution $u(t,x), \beta(t)$ to (2.1) exists and is unique for for $0 \leq t < \tau := \lim_{L\to\infty} \tau^L$*

where $\tau^L := \inf\{t \in \mathbb{R}_+ \cup \{0\} : |\dot{\beta}(t)| \geq L\}$, *and* $\tilde{u}(t,x) := u(t, x + \beta(t))$ *satisfies*

$$\begin{aligned}\tilde{u}(t,x) = &\int_0^\infty p(t,x,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty \dot{\beta}(s)q(t-s,x,y)\tilde{u}(s,y)dyds \\ &+ \int_0^t \int_0^\infty p(t-s,x,y)u(s,y)d\xi_t(x)ds,\end{aligned} \tag{2.2}$$

where p, q *are standard kernels defined in [7].*

2.2 A Transformation

We make the natural transformation $\tilde{u}(t,x) := u(t, x + \beta(t))$, which transforms the original weak definition and its equivalent evolution equation to a nonlinear PDE in the fixed domain $[0,T] \times [0,\infty)$. Then we have

Lemma 2.2.1 *The weak solution* $u(t,x)$ *is obtained by getting it from* $\tilde{u}(t,x)$ *of* (2.2) *by the transformation* $u(t,x) := \tilde{u}(t, x - \beta(t))$, *where*

$$\dot{\beta}(t) = \frac{1}{\rho}\frac{\partial \tilde{u}}{\partial x}(t, 0+).$$

This technique is also used in Chapter 4.

2.3 Existence and Uniqueness of the Truncated Solution

First, in order to control the nonlinear drift term we shall truncate the H^2-norm of the solution (which is shown as equivalent to truncating the nonlinear term, or $\dot{\beta}(t)$), and work with the truncated solution $\tilde{u}^L(t,x)$, namely, the solution to

$$\begin{aligned}\tilde{u}^L(t,x) = &\int_0^\infty p(t,x,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty \dot{\beta}(s)q(t-s,x,y)\Psi_L\left(\|\tilde{u}^L(s,\cdot)\|_H\right)\tilde{u}^L(s,y)dyds \\ &+ \int_0^t \int_0^\infty p(t-s,x,y)u^L(s,y)d\xi_t(x)ds,\end{aligned} \tag{2.3}$$

where $\Psi_L : [0,\infty) \to [0,1]$ is as a smooth monotone decreasing function that satisfies $\chi_{[0,L]} \leq \Psi_L \leq \chi_{[0,L+1]}$.

The existence and uniqueness of the truncated solution is proved by a Picard-type iteration on H^2-spaced combined with similar calculations in Lemma 3.3 of [10]. Unlike in Chapter 3 and Chapter 4, in this problem we need not worry about the existence of the spatial derivatives, or in particular,

$$\frac{\partial \tilde{u}}{\partial x}(t, 0+),$$

which is the right hand side in the Stefan boundary condition, because the stochastic perturbation is smooth in space. By calculations of such an iteration and the structural results about H^2-space (see Section 5.1 of [7]), combined with Lemma 3.3 of [10], we have

Lemma 2.3.1 *Fix $L > 0$. Then the solution $\tilde{u}^L(t, x)$ to (2.3) exists and is unique.*

2.4 Relaxation of the Truncation

Define the stopping time

$$\tau^L := \inf\{t \geq 0 : \|\tilde{u}^L(t, \cdot)\|_H \geq L\}.$$

Also, define

$$\tau := \lim_{L \to \infty} \tau^L,$$

and

$$\tilde{u}(t, x) := \lim_{L \to \infty} \tilde{u}^L(t, x).$$

We then have

Lemma 2.4.1

$$\overline{\lim_{t \nearrow \tau}} \|\tilde{u}(t, \cdot)\|_H = \infty$$

and

$$\overline{\lim_{t \nearrow \tau}} \left| \frac{\partial \tilde{u}}{\partial x}(t, 0+) \right| = \infty.$$

Finally, we have

Lemma 2.4.2 *The solution $\tilde{u}(t, x)$ to (2.2) exists and is unique.*

Combining all the lemmas, we showed the main theorem. Note that the idea of first stopping $|\dot{\beta}(t)|$ from growing too large (that is, exceeding a fixed L), then using this to control the nonlinear drift term, and finally relaxing this truncation and obtaining a global stopping time τ is important, and is also used in Chapter 4 when we study a stochastic Stefan problem driven by a scaled space-time Brownian noise.

Chapter 3

Boundary Regularity of the Stochastic Heat Equation

In the previous chapter we have shown the existence and uniqueness of a stochastic Stefan problem with a spatially-colored and Brownian-in-time noise. Since we would further study a stochastic Stefan problem with space-time Brownian noise under certain regularity conditions, it is necessary that we first understand the effect of such noise on the regularity of the boundary, namely, the differentiability of the solution at the moving boundary, in addition to the existence and uniqueness of the solution itself. This task is critical because the Stefan boundary condition of such a problem involves the spatial derivative of the solution at the boundary, and since the noise is Brownian in space (as well as in time), the solution in general may not have a spatial derivative everywhere except at the boundary.

Therefore, to simplify the problem, we first in this chapter consider a stochastic heat equation of u driven by a multiplicative space-time Brownian noise, so that the noise vanishes at the boundary (where $u = 0$), and we would expect that u is differentiable just at the boundary. Specifically, by removing the shift effect of the moving boundary and studying a stochastic heat equation of this kind, we look to understand

(1) under what sense (or, in what normed space) the solution exists, and the connection between such a norm or space and the differentiability of the solution at the moving boundary;

(2) in what sense ($\mathbb{P}$-a.s.? in L^p? etc.) the Stefan boundary condition holds;

(3) the spatial regularity of the solution (Hölder continuity? with what parameters?) which may guide us on the study of a Stefan problem with a space-time Brownian noise in the next chapter, in particular, the effect of a boundary shift on the iteration of the Itô integral.

3.1 Problem and Main Theorem

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose $W : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a standard 2-dimensional Brownian sheet. Consider the (formal) stochastic heat equation of $u(t, x)$ with a multiplicative space-time Brownian noise on $[0,T]\times[0,\infty)$, under a Dirichlet boundary condition:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial^2 W}{\partial t\partial x}, \forall x>0, t\in[0,T],\\
&u(0,x) = u_0(x),\\
&u(t,0) = 0, \forall x\le 0, t\in[0,T].
\end{aligned} \tag{3.1}$$

From the classic work of [10] by Walsh, the weak solution to the formal equation (3.1) is equivalent to the evolution equation

$$u(t,x) = \int_0^\infty p(t,x,y)u_0(y)dy + \int_0^t\int_0^\infty p(t-s,x,y)u(s,y)W(dyds)$$

where $t\in[0,T], x\ge 0$ and $p(t,x,y)$ is the corresponding kernel as defined in the previous chapter. Then we have the following theorem as the main conclusion of this chapter:

Theorem 3.1.1 *(1) The solution $u(t,x)$ to* (3.1) *exists and is unique with respect to a normed space;*

(2) $\mathbb{P}$-a.s., for all $t\in[0,T]$, $u(t,x)$ is differentiable at $x=0$, and

$$\frac{\partial u}{\partial x}(t,0) = \lim_{x\searrow 0}\frac{u(t,x)}{x} = \int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u_0(y)dy + \int_0^t\int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u(s,y)W(dyds);$$

(3) For $t\in[0,T]$, define $v(t,x) := u(t,x)/x$ for $x>0$ and $v(t,0) := \lim_{x\searrow 0} v(t,x)$, then $\mathbb{P}$-a.s., $v(t,x)$ is $\left(\frac{1}{4}-\epsilon, \frac{1}{6}-\epsilon\right)$-Hölder continuous on $[0,T]\times[0,1]$ for $\epsilon>0$.

Theorem 3.1.1 is proved in Section 3.3 using the integral regularities of a newly defined kernel $\tilde{p}(t,x,y)$ in Section 3.2 combined with a newly defined norm and an argument based on Kolmogorov's Continuity Theorem (see [4]) and Lemma 3.3 of [10] in Section 3.3.

3.2 Integral Regularities of Kernel $\tilde{p}(t,x,y)$

The existence, uniqueness, and regularity of the solution described in Theorem 3.1.1 are based on a number of integral regularities of a newly defined kernel $\tilde{p}(t,x,y)$. We present and prove them in this section.

Define a new kernel $\tilde{p}(t,x,y)$ as

$$\tilde{p}(t,x,y) := \frac{y}{x}p(t,x,y), \forall x>0$$

and

$$\tilde{p}(t,0,y) := \lim_{x\searrow 0} \tilde{p}(t,x,y) = y\frac{\partial p}{\partial x}(t,0,y),$$

then we have

Lemma 3.2.1 *(1)* $\forall x \geq 0$,

$$\int_0^\infty \tilde{p}^2(s,x,y)dy \leq \frac{C}{\sqrt{s}};$$

(2) $\forall x, y \in [0,1], t \in [0,T]$,

$$\int_0^t \int_0^\infty \left[\tilde{p}(s,x,z) - \tilde{p}(s,y,z)\right]^2 dzds \leq C_T(x-y)^{\frac{1}{3}};$$

(3) $\forall x \geq 0, s,t \in [0,T]$,

$$\int_s^t \int_0^\infty \tilde{p}^2(r,x,y)dydr + \int_0^s \int_0^\infty \left[\tilde{p}(r+(t-s),x,y) - \tilde{p}(r,x,y)\right]^2 dydr \leq D_T|t-s|^{\frac{1}{2}}.$$

Proof We only need to prove the above facts for $[0,T]\times(0,1]$, since by Fatou's Lemma they can be extended to the cases for $x, y = 0$.

Throughout the calculations we repeatedly use the following facts:

(a)

$$\int_0^\infty y^n \exp\left(-\frac{y^2}{s}\right) dy = \Gamma\left(\frac{n+1}{2}\right) s^{\frac{n+1}{2}};$$

(b) if f and g are even, then

$$\begin{aligned}&\int_0^\infty g(y)[f(y-x)+f(y+x)]dy = \frac{1}{2}\int_{-\infty}^\infty g(y)[f(y-x)+f(y+x)]dy \\ &= \frac{1}{2}\int_{-\infty}^\infty [g(y-x)+g(y+x)]f(y)dy = \int_0^\infty [g(y-x)+g(y+x)]f(y)dy;\end{aligned}$$

(c) using the fact that $1-\exp(-x) \leq 1 \wedge x$, we have for $A > 0$,

$$\int_0^t \frac{1-\exp\left(-\frac{A}{s}\right)}{\sqrt{s}} ds \leq \int_0^A \frac{ds}{\sqrt{s}} ds + \int_A^\infty \frac{Ads}{s\sqrt{s}} = 4\sqrt{A}.$$

Then

(1)

$$\begin{aligned}\int_0^\infty \tilde{p}^2(s,x,y)dy &= \frac{C_1}{x^2 s}\int_0^\infty y^2\left[e^{-\frac{(x+y)^2}{s}} + e^{-\frac{(x-y)^2}{s}} - 2e^{-\frac{x^2+y^2}{s}}\right]dy\\ &= \frac{C_1}{x^2 s}\int_0^\infty \left[2(x^2+y^2)e^{-\frac{y^2}{s}} - 2y^2 e^{-\frac{x^2+y^2}{s}}\right]dy\\ &= \frac{2C_1}{s}\int_0^\infty e^{-\frac{y^2}{s}}dy + \frac{2C_1}{x^2 s}\left(1-e^{-\frac{x^2}{s}}\right)\int_0^\infty y^2 e^{-\frac{y^2}{s}}dy\\ &\le \frac{2C_1\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} + \frac{2C_1\Gamma\left(\frac{3}{2}\right)}{\sqrt{s}} = \frac{3C_1\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}.\end{aligned}$$

(2) Suppose $x \le y$, and define $h := y - x$. Then from above,

$$\int_0^\infty \left[\tilde{p}(s,x,z) - \tilde{p}(s,y,z)\right]^2 dz = \frac{C_1\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\left[4 + s\left(\frac{1-e^{-\frac{x^2}{s}}}{x^2} + \frac{1-e^{-\frac{(x+h)^2}{s}}}{(x+h)^2}\right)\right] - I_X$$

where

$$\begin{aligned}I_X &= \frac{2C_1}{x(x+h)s}\int_0^\infty z^2\left(e^{-\frac{(x-z)^2}{2s}} - e^{-\frac{(x+z)^2}{2s}}\right)\left(e^{-\frac{(x+h-z)^2}{2s}} - e^{-\frac{(x+h+z)^2}{2s}}\right)dz\\ &= \frac{2C_1}{x(x+h)s}\int_0^\infty z^2\left[e^{-\frac{h^2}{4s}}\left(e^{-\frac{(x+\frac{h}{2}-z)^2}{s}} + e^{-\frac{(x+\frac{h}{2}+z)^2}{s}}\right) - e^{-\frac{\left(x+\frac{h}{2}\right)^2}{s}}\left(e^{-\frac{\left(z+\frac{h}{2}\right)^2}{s}} + e^{-\frac{\left(z-\frac{h}{2}\right)^2}{s}}\right)\right]dz\\ &= \frac{4C_1}{x(x+h)s}\left\{e^{-\frac{h^2}{4s}}\int_0^\infty\left[z^2 + \left(x+\frac{h}{2}\right)^2\right]e^{-\frac{z^2}{s}}dz - e^{-\frac{\left(x+\frac{h}{2}\right)^2}{s}}\int_0^\infty\left[z^2 + \frac{h^2}{4}\right]e^{-\frac{z^2}{s}}dz\right\}\\ &= \frac{2C_1\Gamma\left(\frac{1}{2}\right)}{x(x+h)\sqrt{s}}\left\{e^{-\frac{h^2}{4s}}\left[s + 2\left(x+\frac{h}{2}\right)^2\right] - e^{-\frac{\left(x+\frac{h}{2}\right)^2}{s}}\left[s + 2\left(\frac{h^2}{4}\right)\right]\right\}\\ &= \frac{C_1\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}e^{-\frac{h^2}{4s}}\left\{4 + \frac{1-e^{-\frac{x(x+h)}{s}}}{x(x+h)}(2s+h^2)\right\} \ge \frac{C_1\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}e^{-\frac{h^2}{4s}}\left\{4 + 2s\frac{1-e^{-\frac{x(x+h)}{s}}}{x(x+h)}\right\}.\end{aligned}$$

Note that from (c) above, we get

$$\int_0^t \frac{1-e^{-\frac{h^2}{4s}}}{\sqrt{s}}ds \le 2h.$$

Then we need to consider

$$J(t,x,h) := \int_0^t\left\{\sqrt{s}\left[\frac{1-e^{-\frac{x^2}{s}}}{x^2} + \frac{1-e^{-\frac{(x+h)^2}{s}}}{(x+h)^2} - 2e^{-\frac{h^2}{4s}}\left(\frac{1-e^{-\frac{x(x+h)}{s}}}{x(x+h)}\right)\right]\right\}ds.$$

Now we use two methods to bound $J(t,x,h)$.

(I) Using the fact that $x - \frac{x^2}{2} \le 1 - \exp(-x) \le x$, we get

$$J(t,x,h) \le \int_0^t \frac{2}{\sqrt{s}}\left(1-e^{-\frac{h^2}{4s}}\right)ds + x(x+h)\int_0^t \frac{e^{-\frac{h^2}{4s}}}{s\sqrt{s}}ds \le 4h + \Gamma\left(\frac{1}{2}\right)\frac{x(x+h)}{h}.$$

(II) Define a function

$$\Phi(t,x) := \int_0^t \frac{1-\exp\left(-\frac{x^2}{s}\right)}{x^2} ds = \frac{t}{x^2} - \int_{\frac{x^2}{t}}^{\infty} u^{-2} e^{-u} du.$$

Then

$$\left|\frac{\partial \Phi}{\partial x}(t,x)\right| = 2t\frac{1-e^{-\frac{x^2}{t}}}{x^3} \le \frac{2}{x}.$$

Then we have

$$\begin{aligned} J(t,x,h) &\le \left[\Phi(t,x) + \Phi(t,x+h) - 2\Phi\left(t, x+\frac{h}{2}\right)\right] + \frac{2}{x(x+h)} \int_0^t \sqrt{s}\left(1-e^{-\frac{h^2}{4s}}\right) ds \\ &\le \frac{h}{2}\left(\frac{2}{x} + \frac{2}{x+\frac{h}{2}}\right) + \frac{h^2\sqrt{t}}{x(x+h)} \le \frac{2h}{x} + \frac{h^2\sqrt{t}}{x(x+h)}. \end{aligned}$$

Combining (I) and (II), we have that for some $C_T > 0$, when $x \le h^{2/3}$, using (I) and we get $\int_0^\infty [\tilde{p}(s,x,z) - \tilde{p}(s,y,z)]^2 dz \le C_T h^{1/3}$; when $x > h^{3/2}$, using (II) and we get $\int_0^\infty [\tilde{p}(s,x,z) - \tilde{p}(s,y,z)]^2 dz \le C_T h^{1/3}$.

(3) Suppose $s \le t$, and define $k := t - s$. The first integral on the left hand side is bounded by $C/\sqrt{k}$ using (1). Now, from (1),

$$\int_0^s \int_0^\infty [\tilde{p}(r+k,x,y) - \tilde{p}(r,x,y)]^2 dy = 3C_1\Gamma\left(\frac{1}{2}\right)\left(\sqrt{s+k} - \sqrt{k} + \sqrt{s}\right) - I$$

where (defining $k' := \frac{rk}{2r+k}$)

$$\begin{aligned} I &= \int_0^s \frac{2C_1}{x^2\sqrt{r(r+k)}} \int_0^\infty y^2 \left[e^{-\frac{(x-y)^2}{2(r+k)}} - e^{-\frac{(x+y)^2}{2(r+k)}}\right]\left[e^{-\frac{(x-y)^2}{2r}} - e^{-\frac{(x+y)^2}{2r}}\right] dydr \\ &= \int_0^s \frac{2C_1}{x^2\sqrt{r(r+k)}} \int_0^\infty y^2 \left[e^{-\frac{(x-y)^2}{r+k'}} + e^{-\frac{(x+y)^2}{r+k'}} - e^{-\frac{x^2}{r+\frac{k}{2}}}\left(e^{-\frac{\left(y-\frac{k}{2r+k}x\right)^2}{r+k'}} + e^{-\frac{\left(y+\frac{k}{2r+k}x\right)^2}{r+k'}}\right)\right] dydr \\ &= \int_0^s \frac{4C_1}{x^2\sqrt{r(r+k)}}\left[\int_0^\infty (x^2+y^2) e^{-\frac{y^2}{r+k'}} dy - e^{-\frac{x^2}{r+\frac{k}{2}}} \int_0^\infty \left(\frac{k^2}{(2r+k)^2}x^2 + y^2\right) e^{-\frac{y^2}{r+k'}} dy\right] dr \\ &= 2C_1\Gamma\left(\frac{1}{2}\right)\int_0^s \frac{dr}{\sqrt{r(r+k)}}\left\{2\left[1 - \frac{k^2}{(2r+k)^2} e^{-\frac{x^2}{r+\frac{k}{2}}}\right]\sqrt{r+k'} + \frac{1-e^{-\frac{x^2}{r+\frac{k}{2}}}}{x^2}(r+k')^{\frac{3}{2}}\right\} \\ &\ge 3C_1\Gamma\left(\frac{1}{2}\right)\int_0^s \frac{1}{r+\frac{k}{2}} \frac{r(r+k)}{\left(r+\frac{k}{2}\right)^{\frac{3}{2}}} dr \ge 3C_1\Gamma\left(\frac{1}{2}\right)\int_0^s \frac{rdr}{(r+k)^{\frac{3}{2}}}. \end{aligned}$$

Therefore we have for some $D_T > 0$

$$\int_0^s \int_0^\infty \left[\tilde{p}(r+k,x,y) - \tilde{p}(r,x,y)\right]^2 dy \leq D_T k^{\frac{1}{2}}.\square$$

3.3 Proof of the Main Theorem

Theorem 3.1.1 is proved in two steps. First we show that the solution to (3.1) exists and is unique in a normed space, where the norm is defined so that in the second step the regularity results of the solution can be derived by using the defined norm and the integral regularities of the kernel $\tilde{p}(s,x,y)$ in Lemma 3.2.1. In other words, the norm we defined in the next part characterizes the essential component from which we derive the desired Hölder continuity of the solution and its differentiability at the boundary, which are shown by using Kolmogorov's Continuity Theorem in the second step.

3.3.1 Existence and Uniqueness of the Solution

Existence and uniqueness of the solution is shown by a Picard-type iteration. Consider the iteration

$$u_{n+1}(t,x) = \int_0^\infty p(t,x,y)u_0(y)dy + \int_0^t \int_0^\infty p(t,x,y)u_n(s,y)W(dyds). \tag{3.2}$$

Or, if we define

$$v_n(t,x) := \frac{u_n(t,x)}{x}, v_0(x) := \frac{u_0(x)}{x},$$

then

$$v_{n+1}(t,x) = \int_0^\infty \tilde{p}(t,x,y)v_0(y)dy + \int_0^t \int_0^\infty \tilde{p}(t,x,y)v_n(s,y)W(dyds). \tag{3.3}$$

Fix $p \geq 1$. Suppose $\{f(x)\}_{x\geq 0}$ is a stochastic process. Define a norm

$$\|f\|_{2p} := \sup_{x>0} \mathbb{E}\left[f^{2p}(x)\right].$$

Then we have

Lemma 3.3.1 *For all $p \geq 1$, the solution $v(t,x)$ to* (3.3) *exists and is unique in the space defined by the norm $\|\cdot\|_{2p}$.*

Proof We have

$$\frac{u_{n+1}(t,x)-u_n(t,x)}{x}=\int_0^t\int_0^\infty\left[\frac{u_n(s,y)-u_{n-1}(s,y)}{y}\right]\tilde{p}(t-s,x,y)W(dyds)$$

or equivalently,

$$v_{n+1}(t,x)-v_n(t,x)=\int_0^t\int_0^\infty\left[v_n(s,y)-v_{n-1}(s,y)\right]\tilde{p}(t-s,x,y)W(dyds).$$

Define $H_n(t):=\|v_n(t,\cdot)-v_{n-1}(t,\cdot)\|_{2p}$. Then we have from Lemma 3.2.1 (1) that

$$\begin{aligned}
H_{n+1}(t)&\le\sup_{x>0}C_p\mathbb{E}\left[\int_0^t\int_0^\infty\left(v_n(s,y)-v_{n-1}(s,y)\right)^2p^2(t-s,x,y)dyds\right]^p\\
&\le\sup_{x>0}C_p\left[\int_0^t\int_0^\infty p^2(t-s,x,y)dyds\right]^{p-1}\int_0^t\int_0^\infty\mathbb{E}\left[\left(v_n(s,y)-v_{n-1}(s,y)\right)^{2p}\right]p^2(t-s,x,y)dyds\\
&\le C_p't^{\frac{p-1}{2}}\int_0^t\frac{H_n(s)}{\sqrt{t-s}}ds\le C_{p,T}\int_0^t\frac{H_n(s)}{\sqrt{t-s}}ds.
\end{aligned}$$

where the first inequality comes from Burkholder inequality, the second inequality comes from Jensen's inequality that for $p\ge 1$,

$$\left(\frac{\int ab}{\int b}\right)^p\le\frac{\int a^pb}{\int b}\Leftrightarrow\left(\int ab\right)^p\le\left(\int b\right)^{p-1}\int a^pb,$$

and the third inequality comes from Lemma 3.2.1 (1). By Lemma 3.3 of [10],

$$\sum_n H_n(t)<\infty$$

and the convergence is uniform on compacts. That is, if $0\le t\le T$, then $\sum_n H_n(t)<C_{p,T}<\infty$ where $C_{p,T}$ is a constant dependent only on T. This gives immediately by Picard-type iteration that the solution $v(t,x)$ exists and is unique with respect to $\|\cdot\|_{2p}$. Moreover,

$$\|v(t,\cdot)\|_{2p}\le\|v_0\|_{2p}+\sum_n H_n(t)<D_{p,T}<\infty.\Box$$

Define $u(t,x):=xv(t,x)$, then $u(t,x)$ is the unique solution to (3.2).

3.3.2 Regularity of the Solution and Differentiability at the Boundary

In this part we shall prove the differentiability of the solution at the boundary by giving a $\mathbb{P}$-a.s. limit or the spatial derivative at the boundary. This is shown by using Kolmogorov's Continuity Theorem combined with the integral regularities shown in the previous parts and the existence of the solution under the $\|\cdot\|_{2p}$ norm. In fact, a stronger result is shown, namely, the solution is $\mathbb{P}$-a.s. Hölder continuous with parameters $\left(\frac{1}{4}-\epsilon, \frac{1}{6}-\epsilon\right)$ on $[0,T]\times[0,1]$ for $\epsilon>0$, which can also be used to evaluate the impact of its spatial regularity on the boundary shift effect required in the stochastic Stefan problems drive by space-time Brownian noise.

Lemma 3.3.2 *$\mathbb{P}$-a.s., for all $t\in[0,T]$ the solution $v(t,x)$ to (3.3) is continuous at $x=0$ and*

$$\frac{\partial u}{\partial x}(t,0)=\lim_{x\searrow 0}v(t,x)=\int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u_0(y)dy+\int_0^t\int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u(s,y)W(dyds).$$

Moreover, for $t\in[0,T]$ define $v(t,0):=\lim_{x\searrow 0}v(t,x)$, then $\mathbb{P}$-a.s., $v(t,x)$ is $\left(\frac{1}{4}-\epsilon,\frac{1}{6}-\epsilon\right)$-Hölder continuous on $[0,T]\times[0,1]$ for $\epsilon>0$.

Proof We only need to consider the Brownian term. Fix $p\geq 1$. For $t\in[0,T], x>0$, define

$$I_1(t,x):=\frac{1}{x}\int_0^t\int_0^\infty u(s,y)p(t-s,x,y)W(dyds)=\int_0^t\int_0^\infty v(s,y)\tilde{p}(t-s,x,y)W(dyds),$$

and

$$I_1(t,0):=\int_0^t\int_0^\infty u(s,y)\frac{\partial p}{\partial x}(t-s,0,y)W(dyds)=\int_0^t\int_0^\infty v(s,y)\tilde{p}(t-s,0,y)W(dyds).$$

Then for $t\in[0,T], 0\leq x,y\leq 1$, we have

$$\begin{aligned}
\mathbb{E}\left[(I_1(t,x)-I_1(t,y))^{2p}\right] &\leq C_p\mathbb{E}\left[\int_0^t\int_0^\infty v^2(s,z)\left[\tilde{p}(t-s,x,z)-\tilde{p}(t-s,y,z)\right]^2dzds\right]^p\\
&\leq C_p\left[\int_0^t\int_0^\infty \left[\tilde{p}(t-s,x,z)-\tilde{p}(t-s,y,z)\right]^2dzds\right]^{p-1}\cdot\\
&\qquad\int_0^t\int_0^\infty \mathbb{E}\left[v^{2p}(s,z)\right]\left[\tilde{p}(t-s,x,z)-\tilde{p}(t-s,y,z)\right]^2dzds\\
&\leq C_pD_{p,T}\left[\int_0^t\int_0^\infty \left[\tilde{p}(t-s,x,z)-\tilde{p}(t-s,y,z)\right]^2dzds\right]^p\\
&\leq C'_{p,T}(x-y)^{\frac{p}{3}}
\end{aligned}$$

where the first inequality comes from Burkholder inequality, the second comes from Jensen's inequality, the third comes from the previous part about uniform boundedness of $\|u(t,\cdot)\|_{2p}$ on $[0,T]$, and the last one comes from Lemma 3.2.1 (2).

Similarly, We also have for $0 < s \leq t \leq T, x \in [0,1]$,

$$\mathbb{E}\left[(I_1(t,x) - I_1(s,x))^{2p}\right] \leq 2^{2p-1}[I_{11}(s,t,x) + I_{12}(s,t,x)]$$

where

$$I_{11}(s,t,x) := \mathbb{E}\left[\left(\int_0^s \int_0^\infty v(r,y)\left[\tilde{p}(t-r,x,y) - q_{s-r}(x,y)\right] W(dydr)\right)^{2p}\right]$$
$$I_{12}(s,t,x) := \mathbb{E}\left[\left(\int_s^t \int_0^\infty v(r,y)\tilde{p}(t-r,x,y)W(dydr)\right)^{2p}\right]$$

by Jensen's inequality $\left(\frac{a+b}{2}\right)^{2p} \leq \frac{a^{2p}+b^{2p}}{2}$. Now,

$$\begin{aligned}
I_{11}(s,t,x) &\leq C_p\mathbb{E}\left[\left(\int_0^s \int_0^\infty v^2(r,y)\left[\tilde{p}(t-r,x,y) - \tilde{p}(s-r,x,y)\right]^2 dydr\right)^p\right] \\
&\leq C_p\left[\int_0^s \int_0^\infty \left[\tilde{p}(t-r,x,y) - \tilde{p}(s-r,x,y)\right]^2 dydr\right]^{p-1} \cdot \\
&\qquad \int_0^s \int_0^\infty \mathbb{E}\left[v^{2p}(r,y)\right]\left[\tilde{p}(t-r,x,y) - \tilde{p}(s-r,x,y)\right]^2 dydr \\
&\leq C_{1,p,T}(t-s)^{\frac{p}{2}}
\end{aligned}$$

using Burkholder's inequality, Jensen's inequality, and Lemma 3.2.1 (3). Similarly, using Burkholder's inequality, Jensen's inequality, and Lemma 3.2.1 (1), we have

$$I_{12}(s,t,x) \leq C_{2,p,T}(t-s)^{\frac{p}{2}}.$$

Summing things up, for all $p \geq 1$, we have for $0 \leq s,t \leq T, 0 \leq x,y \leq 1$,

$$\mathbb{E}\left[(I_1(t,x) - I_1(s,y))^{2p}\right] \leq M_{p,T}(x-y)^{\frac{p}{3}} + N_{p,T}(t-s)^{\frac{p}{2}}.$$

By Kolmogorov Continuity Theorem, the above calculations imply that $\mathbb{P}$ almost surely, $I_1(t,x)$ is (γ,β)-Hölder continuous on $[0,T] \times [0,1]$ for all $\gamma < 1/4, \beta < 1/6$. This implies that $I_1(t,0) = \lim_{x \searrow 0} I_1(t,x)$, and that $I_1(t,0)$ is γ-Holder continuous on $[0,T]$ for all $\gamma < 1/4$. $\square$

Chapter 4

A Stochastic Stefan Problem with Space-Time Brownian Noise

In the previous chapter we considered the stochastic heat equation of u driven by a multiplicative space-time Brownian noise $u\frac{\partial^2 W}{\partial t \partial x}$ and showed the existence and uniqueness of the solution under normed space defined by $\|\cdot\|_{2p}$ for all $p \geq 1$ and the $\mathbb{P}$-a.s. differentiability at the boundary and as a by-product, the Hölder continuity of u. The proof provides us with important hints on how to prove the existence and uniqueness of a stochastic Stefan problem driven by space-time Brownian noise, since we shall also prove that the Stefan boundary condition holds simultaneously at the moving boundary.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose $W : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the standard 2-dimensional Brownian sheet. Then the stochastic Stefan problem of $u(t,x)$ has the general (formal) form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}, \forall x > \beta(t),\\
u(t,x) &= 0, \forall x \leq \beta(t),\\
u(0,x) &= u_0(x),\\
\rho\dot{\beta}(t) &= \lim_{x \searrow \beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned}$$

for $0 \leq t < \tau$ where τ is some well defined stopping time.

The introduction of a moving boundary brings about a number of challenges that do not exist in the stochastic heat equation problem, and one of them is to control the boundary shift effect, namely, if we iterate on the boundary $\beta_n(t)$, then $d_n(t) := \beta_{n+1}(t) - \beta_n(t)$ is the shift of the boundary between iterations. We look to control

$$\int_0^t \int_0^\infty \left[\tilde{p}(t-s,x,y+d_n(s))\frac{\sigma(s,y+d_n(s),u(s,y+d_n(s)))}{y+d_n(s)} - \tilde{p}(t-s,x,y)\frac{\sigma(s,y,u(s,y))}{y}\right] W(dyds)$$

whose variance is

$$\int_0^t \int_0^\infty \left[\tilde{p}(t-s,x,y+d_n(s))\frac{\sigma(s,y+d_n(s),u(s,y+d_n(s)))}{y+d_n(s)} - \tilde{p}(t-s,x,y)\frac{\sigma(s,y,u(s,y))}{y}\right]^2 dyds.$$

For multiplicative noise where $\sigma(t,x,u) = u$, the variance above in terms of d_n is completely determined by the spatial regularity of u or $v := u/x$. From the previous chapter we know that spatially v is continuous only at the boundary and has $\left(\frac{1}{6} - \epsilon\right)$-Hölder continuity for $\epsilon > 0$ on $[0,1]$. To evaluate the impact of the spatial regularity on the above variance in terms of d_n, we let $\sigma(t,x,u) := x$, and a calculation with two methods similar to Lemma 3.2.1 (2) shows that

$$\int_0^t \int_0^\infty \left[\tilde{p}(t-s,x,y+d) - \tilde{p}(t-s,x,y)\right]^2 dyds \leq C'_T d^{\frac{2}{3}}.$$

Although the tightness of the inequality is not justified, it still gives strong evidence that it would be difficult to have an iteration on d_n, even if σ is as smooth as $\sigma = x$. Therefore, instead of working with a multiplicative noise, we in this chapter consider a stochastic Stefan problem drive by scaled space-time Brownian noise $\sigma(x)\frac{\partial^2 W}{\partial t \partial x}$ where $\sigma(x)$ satisfies certain regularity conditions:

Definition A function $\sigma : [0,\infty) \to \mathbb{R}$ is a *regular scaling function* if $\sigma(x)$ is Lipschitz in x and $\sigma(x) \sim Ax^\alpha$ at 0 with $\alpha > \frac{3}{2}$. Equivalently, $|\sigma(x)| \leq \min\{Ax^\alpha, Bx\}$ for some $A, B > 0$.

Note that Ax^α provides enough smoothness to control the boundary shift effect as iterating on $\dot{\beta}_n(t)$ below and Bx provides enough control as $x \to \infty$ for other parts so that the noise does not grow too large at infinity.

4.1 Problem and Main Theorem

Suppose $\sigma : [0,\infty) \to \mathbb{R}$ is a regular scaling function. Then we consider the stochastic Stefan problem of $u(t,x)$

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + \sigma(x - \beta(t))\frac{\partial^2 W}{\partial t \partial x}, \forall x > \beta(t), \\
u(t,x) &= 0, \forall x \leq \beta(t), \\
u(0,x) &= u_0(x), \\
\rho\dot{\beta}(t) &= \lim_{x \searrow \beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{4.1}$$

for $0 \leq t < \tau$ for some well defined stopping time τ.

As before we work on a weak formulation of (4.1) and its transformed equivalent evolution equation. Unlike in the colored noise case, we do not directly have the differentiability of the solution at the boundary, so in the iteration of the boundary $\dot{\beta}$ we cannot simply let it be the spatial derivative of the solution at the boundary, but the one similar to what was obtained in the stochastic heat equation, and finally we have an extra step to show that the two coincide $\mathbb{P}$-a.s., or equivalently, the Stefan boundary condition in (4.1) holds.

Theorem 4.1.1 *The solution $u(t,x), \beta(t)$ to (4.1) exists and is unique for $0 \le t < \tau := \lim_{L\to\infty} \tau^L$ where $\tau^L := \inf\{t \in \mathbb{R}_+ \cup \{0\} : |\dot{\beta}(t)| \ge L\}$. Moreover, $\mathbb{P}$-a.s., $\beta \in C^1([0,\tau))$ and $\dot{\beta}(t)$ is $\left(\frac{1}{4}-\epsilon\right)$-Hölder continuous for $\epsilon > 0$.*

Theorem 4.1.1 is proved in Section 4.3 using the integral regularities of the new kernels $\tilde{p}(t,x,y)$ and $\tilde{q}(t,x,y)$ proved in Section 3.2 and Section 4.2. The regularity result on the moving boundary is proved via using Kolmorogov's Continuity Theorem similarly to the proof given in Section 3.3.

4.2 Integral Regularities of Kernels $\tilde{p}(t,x,y), \tilde{q}(t,x,y)$

As in the previous chapter, the proof to the main theorem is based on the integral regularities of newly defined kernels. Define two new kernels (where $\tilde{p}$ is the same as in the previous chapter)

$$\tilde{p}(t,x,y) := \frac{y}{x}p(t,x,y), \tilde{p}(t,0,y) := \lim_{x\searrow 0}\tilde{p}(t,x,y) = y\frac{\partial p}{\partial x}(t,0,y) = \frac{y^2}{t\sqrt{t}}\exp\left[-\frac{y^2}{2t}\right];$$
$$\tilde{q}(t,x,y) := \frac{y}{x}\frac{\partial p}{\partial x}(t,x,y), \tilde{q}(t,0,y) := \lim_{x\searrow 0}\tilde{q}(t,x,y) = y\frac{\partial^2 p}{\partial x^2}(t,0,y) = 0.$$

Then we have the following integral regularity results about $\tilde{q}(t,x,y)$, similar to those for $\tilde{p}(t,x,y)$ as in Lemma 3.2.1:

Lemma 4.2.1 *There exists $C > 0, K > 0$ such that*

(1)
$$\int_0^\infty |\tilde{q}(s,x,y)|dy < \frac{C}{\sqrt{s}};$$

(2) for $x, x+h \in [0,1]$,
$$\int_0^t\int_0^\infty |\tilde{q}(t-s,x+h,y) - \tilde{q}(t-s,x,y)|dyds < Kh^{\frac{1}{6}};$$

(3) for $k \ge 0$,
$$\int_0^t\int_0^\infty |\tilde{q}(t+k-s,x,y) - \tilde{q}(t-s,x,y)|dyds < Kk^{\frac{1}{4}}.$$

Proof First we observe that

$$\tilde{q}(t,x,y) = \frac{y^2}{xs}p_+(t,x,y) - \frac{y}{s}p(t,x,y) = \frac{y^2}{xs}p_+(t,x,y) - \frac{x}{s}\tilde{p}(t,x,y)$$

where

$$p_+(t,x,y) := \frac{C_1}{\sqrt{s}}\left[e^{-\frac{(x-y)^2}{2s}} + e^{-\frac{(x+y)^2}{2s}}\right].$$

Therefore the calculations to prove the above facts are fairly similar to those in Lemma 3.2.1, hence we only provide sketch here.

(1) this can be shown by using facts (a) and (b) in Lemma 3.2.1, where

$$g(y) := |y|,$$
$$f(y) := |y| \exp\left(-\frac{y^2}{2s}\right),$$

followed by a direct calculation of the integral after applying (b) with f, g.

(2) Similar to Lemma 3.2.1 (2), we still use a two-method approach to estimate the first part containing p_+:

$$\int_0^t \int_0^\infty \frac{y^4}{s^2}\left[\frac{p_+(t-s,x+h,y)}{x+h} - \frac{p_+(t-s,x,y)}{x}\right]^2 dyds \le Ch^{\frac{1}{3}}.$$

From Lemma 3.2.1 (1) and (c) we also have

$$\int_0^t \frac{ds}{s}\int_0^\infty |(x+h)\tilde{p}(t-s,x+h,y) - x\tilde{p}(t-s,x,y)|\, dy \le C'h.$$

Combining the above two facts, we have there exists $K > 0$ such that

$$\int_0^t \int_0^\infty |\tilde{q}(t-s,x+h,y) - \tilde{q}(t-s,x,y)| dyds < Kh^{\frac{1}{6}}.$$

(3) Similar to Lemma 3.2.1 (3), we first estimate the first part containing p_+:

$$\int_0^t \int_0^\infty \frac{y^4}{x^2s^2}\left[p_+(t+k-s,x,y) - p_+(t-s,x,y)\right]^2 dyds \le Ck^{\frac{1}{2}}.$$

From Lemma 3.2.1 (1) and (c) we also have

$$\int_0^t \frac{xds}{s}\int_0^\infty |\tilde{p}(t+k-s,x+h,y) - \tilde{p}(t-s,x,y)|\, dy \le C'k^{\frac{1}{2}}.$$

Combining the above two facts, we have there exists $K > 0$ such that

$$\int_0^t \int_0^\infty |\tilde{q}(t+k-s,x,y) - \tilde{q}(t-s,x,y)| dyds < Kk^{\frac{1}{4}}. \square$$

4.3 Proof of the Main Theorem

As in the previous chapters, we work with the weak formulation of the solution and its equivalent evolution equation. Define $\tilde{u}(t,x) := u(t, x+\beta(t))$, Then the equivalent evolution equation gives

$$\begin{aligned} \tilde{u}(t,x) = &\int_0^\infty p(t,x,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty \dot{\beta}(s)q(t-s,x,y)\tilde{u}(s,y)dyds \\ &+ \int_0^t \int_0^\infty p(t-s,x,y)\sigma(y)W_\beta(dyds), \end{aligned} \tag{4.2}$$

where W_β is the Brownian sheet obtained by shifting W spatially by $x \to x + \beta(t)$. The main theorem is proved by adopting the following strategy:

(1) prove the existence and uniqueness of the truncated solution $\tilde{u}^L(t,x)$ and its corresponding $\beta(t)$ in $\|\cdot\|_{2p}$ for $0 \le t < \tau^L$ for a fixed $L > 0$ by an argument based on Picard-type iterations;

(2) based on (1), show that $\mathbb{P}$-a.s., the Stefan boundary condition

$$\lim_{x \searrow 0} \frac{\tilde{u}^L(t,x)}{x} = \rho\dot{\beta}(t)$$

holds by using Kolmogorov's Continuity Theorem on the drift term and the Brownian term;

(3) let $L \to \infty$ and obtain the solution $\tilde{u}(t,x)$; transform $\tilde{u}$ back to u and obtain a weak solution of (4.1).

4.3.1 Existence and Uniqueness of the Truncated Solution

In this section we first show the existence and uniqueness of β that satisfies (4.3), and then truncate $\dot{\beta}(t)$ by a fixed $L > 0$ so that the drift term of the solution is controlled from growing too large. Then we show the existence and uniqueness of the solution under the truncation.

Defining W_β as before, then we have

$$\begin{aligned}\rho\dot{\beta}(t) = &\int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty \frac{\partial p}{\partial x}(t-s,0,y)\sigma(y)W_\beta(dyds),\end{aligned} \tag{4.3}$$

Lemma 4.3.1 *There exists a unique* $\beta(t)$ *that satisfies* (4.3).

Proof Consider the following iteration on $\beta(t)$:

$$\begin{aligned}\rho\dot{\beta}_n(t) = &\int_0^\infty \frac{\partial p_-}{\partial x}(t,0,y)u_0(y)dy \\ &+ \int_0^t \int_0^\infty \tilde{p}(t-s,0,y)\frac{\sigma(y)}{y}W_{\beta_{n-1}}(dyds).\end{aligned}$$

Fix $p \geq 1$ and define

$$H_n(t) := \mathbb{E}\left[(\dot{\beta}_{n+1}(t) - \dot{\beta}_n(t))^{2p}\right].$$

Also define

$$\mathbf{g}(s,y) := \tilde{p}(s,0,y)y^{\alpha-1}.$$

Then letting $d_n(t) := \beta_n(t) - \beta_{n-1}(t)$, we have

$$\begin{aligned}H_n(t) &\leq K_1\mathbb{E}\left\{\int_0^t\int_0^\infty \left[\mathbf{g}(t-s,y+|d_n(s)|) - \mathbf{g}(t-s,y)\right]^2 dyds\right\}^p \\ &\leq K_1\mathbb{E}\left\{\int_0^t |d_n(s)|^2 \int_0^\infty \left[\int_0^1 \frac{\partial \mathbf{g}}{\partial y}(t-s,y+\lambda|d_n(s)|)d\lambda\right]^2 dyds\right\}^p \\ &\leq K_2\mathbb{E}\left\{\int_0^1\int_0^t |d_n(s)|^2 \int_0^\infty \frac{(y+\lambda|d_n(s)|)^{2\alpha}}{(t-s)^3}\exp\left[-\frac{(y+\lambda|d_n(s)|)^2}{(t-s)}\right] dydsd\lambda\right\}^p \\ &\quad + K_3\mathbb{E}\left\{\int_0^1\int_0^t |d_n(s)|^2 \int_0^\infty \frac{(y+\lambda|d_n(s)|)^{2\alpha+4}}{(t-s)^5}\exp\left[-\frac{(y+\lambda|d_n(s)|)^2}{(t-s)}\right] dydsd\lambda\right\}^p \\ &\leq K_4\mathbb{E}\left\{\int_0^t \frac{|d_n(s)|^2}{(t-s)^{\frac{5}{2}-\alpha}}ds\right\}^p \leq K_5\mathbb{E}\left\{\int_0^t\int_0^s \frac{s(\beta_n(r)-\beta_{n-1}(r))^2}{(t-s)^{\frac{5}{2}-\alpha}}drds\right\}^p \\ &= K_5\mathbb{E}\left\{\int_0^t\int_r^t \frac{s(\beta_n(r)-\beta_{n-1}(r))^2}{(t-s)^{\frac{5}{2}-\alpha}}dsdr\right\}^p.\end{aligned}$$

Note that for the above argument to work we must have $\alpha > \frac{3}{2}$, otherwise the first integral with respect to r is ∞. Since $\frac{5}{2} - \alpha > -1$, we get that

$$H_n(t) \leq K\int_0^t H_{n-1}(s)(t-s)^{\frac{3}{2}-\alpha}ds.$$

Also, define

$$\rho\dot{\beta}_0(t) = \int_0^\infty \frac{\partial p_-}{\partial x}(t,0,y)u_0(y)dy + \int_0^t\int_0^\infty \tilde{p}(t-s,0,y)\frac{\sigma(y)}{y}W(dyds),$$

then from Lemma 3.2.1 (1) we have

$$\mathbb{E}|\dot{\beta}_0(t)|^{2p} < \infty$$

Therefore, by Walsh's Lemma 3.3 in [10], we obtain that $\beta(t)$ as the limit of $\beta_n(t)$ exists and is unique, and we also have

$$\mathbb{E}|\dot{\beta}(t)|^{2p} < \infty$$

and its bound is uniform (that is, not dependent of t). □

Let $\dot{\beta}(t)$ be the solution to (4.3). Fix $L > 0$, and define $\Psi_L : [0,\infty) \to [0,1]$ as a smooth monotone decreasing function that satisfies $\chi_{[0,L]} \le \Psi_L \le \chi_{[0,L+1]}$. Now consider the following iteration of $\tilde{u}^L(t,x)$:

$$\begin{aligned}\tilde{u}_n^L(t,x) = &\int_0^\infty p_-(t,x,y)u_0(y)dy \\ &+ \int_0^t\int_0^\infty \dot{\beta}(s)\frac{\partial p_-}{\partial x}(t-s,x,y)\tilde{u}_{n-1}^L(s,y)\Psi_L(|\dot{\beta}(s)|)dyds \\ &+ \int_0^t\int_0^\infty p_-(t-s,x,y)\sigma(y)W_\beta(dyds),\end{aligned} \tag{4.4}$$

where the initial value

$$\tilde{u}_0^L(t,x) := \int_0^\infty p_-(t,x,y)u_0(y)dy + \int_0^t\int_0^\infty p_-(t-s,x,y)\sigma(y)W_\beta(dyds).$$

Then we have

Lemma 4.3.2 *The solution $\tilde{u}^L(t,x)$ to (4.4) exists and is unique.*

Proof By adopting a similar approach to the proof of Theorem 3.1.1, we first compute

$$\frac{\tilde{u}_{n+1}^L(t,x) - \tilde{u}_n^L(t,x)}{x} = \int_0^t\int_0^\infty \dot{\beta}(s)\tilde{q}(t-s,x,y)\left[\frac{\tilde{u}_n^L(s,y)-\tilde{u}_{n-1}^L(s,y)}{y}\right]\Psi_L(|\dot{\beta}(s)|)dyds.$$

Define

$$J_n(t) := \sup_{x>0}\mathbb{E}\left[\left|\frac{\tilde{u}_{n+1}^L(t,x)-\tilde{u}_n^L(t,x)}{x}\right|^{2p}\right].$$

From Lemma 4.2.1 (2), still use Burkholder's inequality and Jensen's inequality and we get

$$\begin{aligned} J_n(t) &\le K_1(L+1)^{2p}\mathbb{E}\left\{\int_0^t\int_0^\infty \tilde{q}(t-s,x,y)\left[\frac{\tilde{u}_n^L(s,y)-\tilde{u}_{n-1}^L(s,y)}{y}\right]dyds\right\}^{2p} \\ &\le K_2\int_0^t \frac{J_{n-1}(s)}{\sqrt{t-s}}ds. \end{aligned}$$

Also, from Lemma 3.2.1 (1) and $|\sigma(y)| \le By$, by following the same calculations as in the previous chapter we get

$$\sup_{x>0}\mathbb{E}\left[\left|\frac{\tilde{u}_0^L(t,x)}{x}\right|^{2p}\right] < \infty$$

Therefore, by Walsh's Lemma 3.3 in [10], we obtain that $\tilde{u}^L(t,x)$ as the limit of $\tilde{u}_n^L(t,x)$ exists and is unique, and we also have

$$\sup_{x>0}\mathbb{E}\left[\left|\frac{\tilde{u}^L(t,x)}{x}\right|^{2p}\right] < \infty$$

and its bound is uniform (that is, not dependent of t). □

4.3.2 The Stefan Boundary Condition Holds

Lemma 4.3.3 *Let $\tilde{u}^L$ and β be the unique solutions to* (4.4) *and* (4.3). *Then $\mathbb{P}$-a.s., the Stefan boundary condition holds, namely,*

$$\lim_{x\searrow 0}\frac{\tilde{u}^L(t,x)}{x} = \rho\dot{\beta}(t), \forall 0\le t<\tau^L$$

and $\dot{\beta}(t)$ is $\left(\frac{1}{4}-\epsilon\right)$-Hölder continuous.

Proof This lemma is shown by using Kolmogorov Continuity Theorem and adopting the similar calculations as in the previous chapter. In fact, define for $x>0$ $v(t,x) := \tilde{u}^L(t,x)/x$ and $v(t,0) := \rho\dot{\beta}(t)$, then we claim that $\mathbb{P}$-a.s., $v(t,x)$ is $\left(\frac{1}{4}-\epsilon, \frac{1}{6}-\epsilon\right)$-Hölder continuous for fixed $T>0$ on $[0,T]\times[0,1]$ with $\epsilon>0$.

Indeed, by Burkholder's inequality,

$$\begin{aligned} \mathbb{E}\left[|v(t,x+h)-v(t,x)|^{2p}\right] &\le K_1\mathbb{E}\left[\int_0^t\int_0^\infty [\tilde{q}(t-s,x+h,y)-\tilde{q}(t-s,x,y)]v(s,y)dyds\right]^{2p} \\ &\quad + K_2\mathbb{E}\left[\int_0^t\int_0^\infty [\tilde{p}(t-s,x+h,y)-\tilde{p}(t-s,x,y)]^2v^2(s,y)dyds\right]^p. \end{aligned}$$

By Lemma 3.2.1 (2) and Lemma 4.2.1 (2) about $\tilde{p}$ and $\tilde{q}$, we have that by using Jensen's inequality as in the previous chapter, for $0\le x\le 1$,

$$\mathbb{E}\left[|v(t,x+h)-v(t,x)|^{2p}\right] \le Kh^{\frac{p}{3}}$$

where K does not depend on t. Also, by Lemma 3.2.1 (3) and Lemma 4.2.1 (3), we have that still by using Jensen's inequality, for $0 \le t \le T$,

$$\mathbb{E}\left[|v(t+k,x) - v(t,x)|^{2p}\right] \le Kk^{\frac{p}{2}}.$$

By Kolmogorov's Continuity Theorem, we have that for all $\delta < 1/4, \gamma < 1/6$, $v(t,x)$ is $\mathbb{P}$-a.s. uniformly (δ, γ)-Hölder continuous, which implies that $\mathbb{P}$-a.s.,

$$\lim_{x \searrow 0} v(t,x) = \rho\dot{\beta}(t), \forall t \in [0,T],$$

and $\dot{\beta}(t) = v(t,0)/\rho$ is $\left(\frac{1}{4} - \epsilon\right)$-Hölder continuous.

Now, when $\tau^L < \infty$, we simply let $T := \tau^L$. When $\tau^L = \infty$, we choose $T := n$ for all $n \in \mathbb{N}$, and since the above statement holds for $t \in [0,n]$ for all $n \in \mathbb{N}$ we have that it holds for $t \in [0, \infty)$. $\square$

4.3.3 Relaxation of the Truncation

Lemma 4.3.4 *Define* $\tau := \lim_{L\to\infty} \tau^L$ *and* $\tilde{u}(t,x) := \lim_{L\to\infty} \tilde{u}^L(t,x)$. *Then* $\tilde{u}(t,x)$ *is the unique solution to* (4.1).

Proof Fix $t \in [0,\tau)$ and define $L_t := \sup\{|\dot{\beta}(r)| : 0 \le r \le t\}$. Then we have $L_t < \infty$. Therefore for $0 \le r \le t$ we have that $\tilde{u}(r,x)$ also satisfies the equation of $\tilde{u}^{L_t}(r,x)$. By the uniqueness of β and $\tilde{u}^{L_t}$, this implies that for $0 \le r \le t$,

$$\tilde{u}(r,x) = \tilde{u}^{L_t}(r,x) \text{ and } \lim_{x \searrow 0} \frac{\tilde{u}(r,x)}{x} = \rho\dot{\beta}(r).$$

Therefore, for $0 \le t < \tau$, $\tilde{u}(t,x)$ satisfies (4.1) and also

$$\lim_{x \searrow 0} \frac{\tilde{u}(t,x)}{x} = \rho\dot{\beta}(t).\square$$

4.4 Numerical Simulation

Since the existence and uniqueness of the problem is shown we can indeed simulate the solution and its boundary numerically. The simulation uses finite-difference Euler approximation scheme described in [2]. To guarantee numerical robustness in our simulation we assume the space and time increment steps satisfy $\Delta t < (\Delta x)^2/2$.

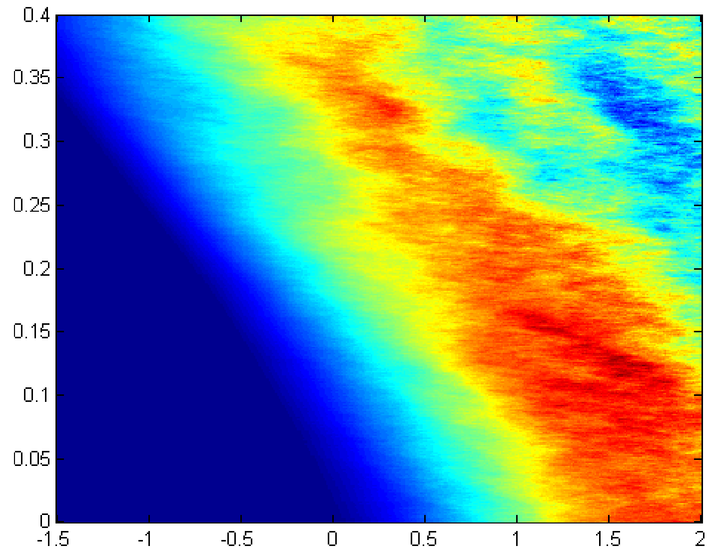

Figure 4.1: Weak solution $u(t,x)$

In this simulation we apply the following simulation parameters:

$$\rho = -0.2;$$
$$u_0(x) = \frac{x+x^2}{1+x^4/16};$$
$$\sigma(x) = \frac{x^2}{1+4x}.$$

The consequent numerical results are illustrated as follows.

(1) Figure 4.1 illustrates the weak solution $u(t,x)$;

(2) Figure 4.2 illustrates the boundary derivative $\dot{\beta}(t)$;

(3) Figure 4.3 illustrates the typical shape of the solution $u(t,x)$ at a particular time $t>0$, from which we see that u is smoother as x gets closer to the boundary (shifted and denoted by 0), and is differentiable at the boundary.

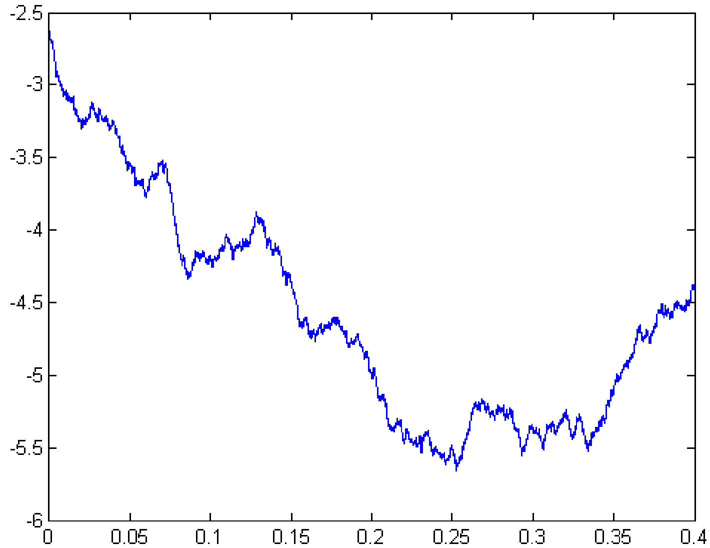

Figure 4.2: Boundary derivative (or speed of moving boundary) $\dot{\beta}(t)$

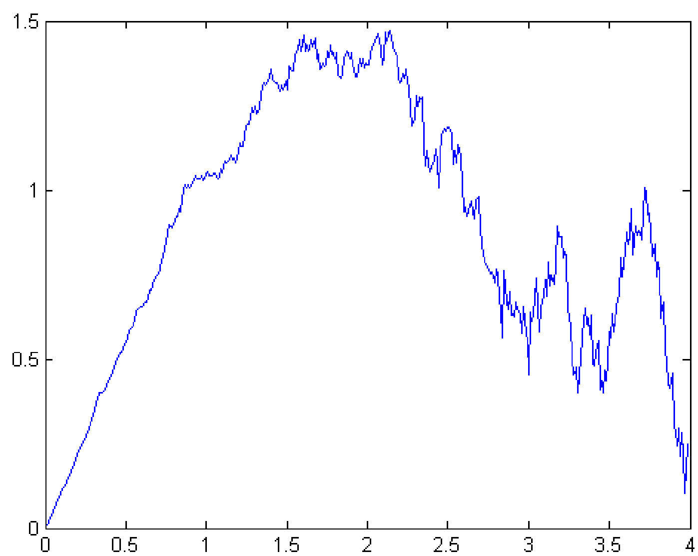

Figure 4.3: A typical shape of the solution u at some time $t > 0$. u is smoother as x gets closer to the boundary (shifted and denoted by 0), and is differentiable at the boundary.

Chapter 5

Market Limit Orders: Modeling, Parameter Estimation, and Optimization

In literature there have been researches on the modeling of market limit orders and their execution such as [8] and [9], most of which are based on discrete settings, for instance, models based on Poisson processes and/or queuing theory. However, because of rapid technological evolution which brings about ultra-fast microprocessors and hardware, trading behaviors and patterns involved with a large amount of limit order creation, transaction, and cancellation within a short period of time, such as high frequency trading (HFT), have become quite popular and tend to have a heavy impact on the mechanisms of price discovery and formulation. In this chapter, consequently, a continuous and dynamic model of market limit orders and their dynamics is proposed, where stochastic Stefan equations are used to describe the evolution of the limit order books of the market. Specifically, the model is based on the following facts:

(1) limit orders are placed, cancelled, and executed in a manner where jitters tend to be rapidly smoothed out, and the volume roughly decreases as the (limit) price goes beyond the mid-price;

(2) the change of the mid-price is driven by the transactions of the ask or bid orders; for instance, the mid-price would drop if more ask orders are created and/or more bid orders are fulfilled;

(3) the randomness of the model comes from the constant creation, cancellation, and transaction of limit orders; moreover, the intensity of such behaviors varies according to different (limit) prices, and tends to vanish as the price goes far beyond the mid.

This chapter elaborates as follows. Section 5.1 describes the model of market limit orders with the preceding facts being described by the Stefan equations accordingly; Section 5.2 discusses the method of estimation of model parameters based on a given limit order dataset, where a Maximum-Likelihood Estimator (MLE) and an estimator that minimizes the Mean-Square Errors (MSE) under AIC (see [1]) are given; Section 5.3 studies the investment optimization problem to maximize the investor's utility function and find the optimal amount-to-buy of the underlying asset and the consumption given a fixed amount of wealth at a given time for an investor who takes the model as exogenous, derives two theorems as static and dynamic criteria for test of optimality from the model dynamics using Itô's Formula, and presents an intuitive interpretation to them.

5.1 Modeling

We model the evolution of the limit orders of a particular asset in the market. Suppose we work within the time interval $[0,T]$, and S denotes the natural log of the price so $S \in \mathbb{R}$. At a particular time point $t \in [0,T]$, suppose the volume of the ask (resp. bid) limit orders on the limit order book from S to $S+dS$ is $V_A(t,S)dS$ (resp. $V_B(t,S)dS$), and suppose the natural log of the mid-price is $S^*(t)$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, We then have the following model assumptions:

(1) all matched orders are executed immediately since all major trading centers today have been computerized, then we have for all (S,t) such that $S > S^*(t)$, $V_B(S,t) = 0$; similarly for all (S,t) such that $S < S^*(t)$, $V_A(S,t) = 0$;

(2) since jitters have a tendency to be rapidly smoothed out, $\partial V_A/\partial t$ contains a component $\alpha_A \Delta V_A$ where α_A is a positive constant and $\Delta := \partial^2/\partial S^2$ is the Laplacian; the same is true for $V_B(S,t)$ with α_B;

(3) the change of mid-price is driven by the "strength" of the ask and bid orders placed around the mid-price, which implies we have a Stefan-type condition

$$\rho \frac{dS^*}{dt}(t) = \left[\frac{\partial V_A}{\partial S}(t, S^*(t)+) + \frac{\partial V_B}{\partial S}(t, S^*(t)-)\right] \tag{5.1}$$

where ρ is a constant;

(4) for $S \geq S^*(t)$ the ask orders are placed in a stochastic manner, hence another component of $\partial V_A/\partial t$ is $\sigma_A(|S - S^*(t)|)\frac{\partial^2 W}{\partial t \partial S}$ where σ_A is a regular scaling function (defined in Chapter 4) and $W : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a standard 2-dimensional Brownian sheet; the same is true for V_B with $S \leq S^*(t)$ and σ_B.

The model is illustrated in Figure 5.1.

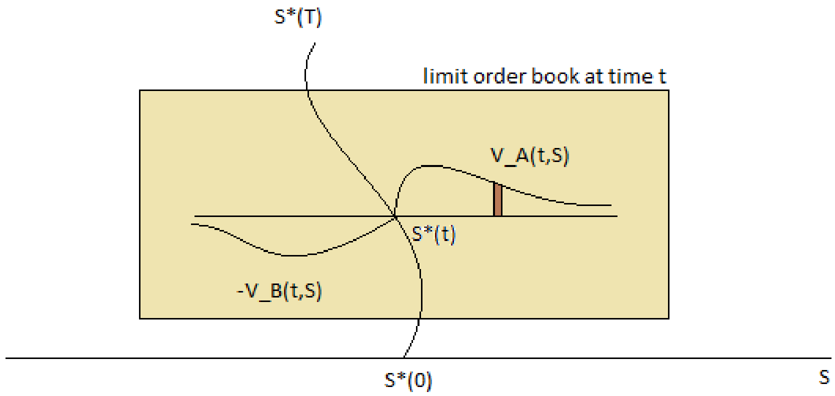

Figure 5.1: The evolution model of the limit order book

In sum, for a set of given model parameters $u_{0A}, u_{0B}, \alpha_A, \alpha_B, \sigma_A, \sigma_B, \rho$, the model has

$$
\begin{aligned}
\frac{\partial V_A}{\partial t} &= \alpha_A \frac{\partial^2 V_A}{\partial S^2} + \sigma_A(|S - S^*(t)|)\frac{\partial^2 W}{\partial t \partial S}, \forall S > S^*(t),\\
V_A(t,S) &= 0, \forall S \leq S^*(t),\\
V_A(0,S) &= u_{0A}(S),\\
\frac{\partial V_B}{\partial t} &= \alpha_B \frac{\partial^2 V_B}{\partial S^2} + \sigma_B(|S - S^*(t)|)\frac{\partial^2 W}{\partial t \partial S}, \forall S < S^*(t),\\
V_B(t,S) &= 0, \forall S \geq S^*(t),\\
V_B(0,S) &= u_{0B}(S),\\
\rho \frac{dS^*}{dt}(t) &= \left[\frac{\partial V_A}{\partial S}(t, S^*(t)+) + \frac{\partial V_B}{\partial S}(t, S^*(t)-)\right].
\end{aligned}
\tag{5.2}
$$

Theorem 5.1.1 *The solution V_A, V_B, S^* to model (5.2) exists and is unique for $0 \leq t < \tau$ where τ is a well-defined stopping time.*

Proof We roughly follow the same procedure to show the existence and uniqueness as in Chapter 4. First we make the following transformation: $\tilde{V}_A(t,S) := V_A(t, S^*(t) + S), \tilde{u}_{0A}(S) := u_{0A}(S^*(0) + S), \tilde{V}_B(t,S) := V_B(t, S^*(t) - S), \tilde{u}_{0B}(S) := u_{0B}(S^*(0) - S)$. Then the same argument as in Theorem 4.1.1 shows that there exists unique β_A, β_B that is the limit of the iteration

$$
\begin{aligned}
\rho \dot{\beta}_A^n(t) &= \int_0^\infty \frac{\partial \tilde{p}}{\partial x}(t,0,y)\tilde{u}_{0A}(y)dy + \int_0^t \int_0^\infty \frac{\partial \tilde{p}}{\partial x}(t-s,0,y)\sigma_A(y)W_{\beta^n}(dyds),\\
\rho \dot{\beta}_B^n(t) &= \int_0^\infty \frac{\partial \tilde{p}}{\partial x}(t,0,y)\tilde{u}_{0B}(y)dy + \int_0^t \int_0^\infty \frac{\partial \tilde{p}}{\partial x}(t-s,0,y)\sigma_B(y)W_{\beta^n}(dyds),\\
\beta^n(t) &= \beta_A^n(t) - \beta_B^n(t).
\end{aligned}
$$

Define $S^*(t) := \lim_n \beta^n(t)$ which is also unique. Then using the same argument as in Theorem 4.1.1, we have that $\tilde{V}_A$ and $\tilde{V}_B$ exist and are unique. Also, using Kolmogorov's Continuity Theorem as in Lemma 4.3.3, we have that almost surely,

$$\rho\dot{\beta}_A(t) = \frac{\partial V_A}{\partial S}(t, S^*(t)+),$$
$$\rho\dot{\beta}_B(t) = -\frac{\partial V_B}{\partial S}(t, S^*(t)-),$$

where $0 \leq t < \tau := \lim_{L\to\infty} \tau^L$, and $\tau^L := \inf\{t \geq 0 : |\dot{\beta}_A(t)| \geq L \text{ or } |\dot{\beta}_B(t)| \geq L\}$. Therefore the Stefan boundary condition (5.1) holds. $\square$

5.2 Parameter Estimation

In this section a statistical method based on AIC is developed to estimate the parameters of model (5.2) given a real dataset of the limit order book and the mid-price of a particular asset within a certain period of time.

Suppose the dataset consists of 3 matrices, two $T \times N$ matrices D_A, D_B for the volumes of the ask and bid orders, and a $T \times 1$ matrix P for the mid-price. The sampling steps for time and price are Δ_T and Δ_N, that is, the dataset is from time 0 to $\Delta_T T$, and the t-th row of D_A stores the volumes of the ask orders from limit price $P[t]$ to $P[t] + \Delta_N N$, while the t-th row of D_B stores the volumes of the bid orders from limit price $P[t] - \Delta_N N$ to $P[t]$, where $[\cdot]$ denotes the vector subscription.

Our goal in this section is to develop an algorithm to numerically compute or even find an explicit expression of the statistics served as the appropriate estimators of the model parameters under maximum-likelihood, minimum mean-square errors, and AIC. The method is decomposed into two parts, first parameters $\alpha_A, \alpha_B, \sigma_A, \sigma_B$ are estimated using maximum-likelihood approach combined with AIC based on the limit order book data D_A and D_B, and then u_{0A}, u_{0B}, ρ are estimated using least mean-square-errors combined with AIC based on the whole dataset D_A, D_B, P.

5.2.1 AIC/MLE Estimation of $\alpha_A, \alpha_B, \sigma_A, \sigma_B$

To estimate σ_A and σ_B, we further assume that

$$\sigma_i(x) := \frac{x^{1.6}}{1 + xp_i(x)}, i = A, B$$

where for $i = A, B$, p_i is a polynomial with $\deg p_i \geq 1$, so that σ_i is a regular scaling function, and $\lim_{x\to\infty} \sigma_i(x) = 0$, because there tends to be very little randomness as the limit price goes far beyond the mid-price. Then the goal is to estimate α_A, α_B and the degrees of p_A, p_B and their coefficients. Since the number of parameters $4 + \deg p_A + \deg p_B$ is also to be estimated, an AIC-based approach is used.

Suppose $i = A, B$, and $p_i(x) = \sum_{j=0}^{d_i} p_{ij} x^j$ with $d_i \geq 1$. To estimate $\alpha_i, p_{i0}, \ldots, p_{id_i}$, we use a finite-difference Euler approximation scheme similar in Chapter 4. For each integer n, define the difference operator $\nabla : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by

$$\nabla \left((a_1, \ldots, a_n)^T \right) = (a_2 - a_1, \ldots, a_n - a_{n-1})^T .$$

Denote $(\nabla a)[i]$ by ∇a_i. For a matrix with row label R and column label C denote ∇_R as the difference operator on column vectors and ∇_C as that on row vectors. Then for each $i = A, B, t = 1, \ldots, T-1, S = 1, \ldots, N-2$,

$$\frac{\nabla_t D_i[t,S]}{\Delta_T \Delta_N} = \alpha_i \frac{\nabla_S^2 D_i[t,S]}{\Delta_N^3} + \sigma_i(S\Delta_N) \frac{\xi_{t,S}}{\Delta_T \Delta_N}$$

where

$$\xi_{t,S} := W\left([\Delta_T t, \Delta_T(t+1)] \times [S^*(\Delta_T t) + \Delta_N S, S^*(\Delta_T t) + \Delta_N(S+1)] \right)$$

are independent identically distributed Gaussian random variables with mean 0 and variance $\Delta_T \Delta_N$. Fix $d_i = \deg p_i$, then the log likelihood function $\ell_i(\alpha_i, p_{i0}, \ldots, p_{id_i})$ satisfies

$$-2\ell_i(\alpha_i, p_{i0}, \ldots, p_{id_i}) = \sum_{t=1}^{T-1} \sum_{S=1}^{N-2} \left\{ \left[\nabla_t D_i[t,S] - \alpha_i \left(\frac{\Delta_T}{\Delta_N^2} \right) \nabla_S^2 D_i[t,S] \right] \frac{1 + \sum_{j=0}^{d_i} p_{ij} S^{j+1} \Delta_N^{j+1}}{S^{1.6} \Delta_N^{1.6}} \right\}^2 . \quad (5.3)$$

When d_i is fixed, the first order condition is a cubic formula of the variables, which shall be solved using numerical methods. If the model can be degenerated so that $\alpha_i = \alpha_{0i}$ is a known constant, then we can indeed solve for optimal $\hat{p}_{i0}, \ldots, \hat{p}_{id_i}$:

$$\mathbf{p}_i = -\mathbf{A}_i^{-1} \mathbf{b}_i,$$

where $\mathbf{p}_i = (\hat{p}_{i0}, \ldots, \hat{p}_{id_i})^T$, $\mathbf{A}_i$ is a $(d_i + 1)$-square matrix with its (m, n) element being

$$\sum_{t=1}^{T-1} \sum_{S=1}^{N-2} \left\{ \left[\nabla_t D_i[t,S] - \alpha_{0i} \left(\frac{\Delta_T}{\Delta_N^2} \right) \nabla_S^2 D_i[t,S] \right] S^{m+n-3.2} \Delta_N^{m+n} \right\}^2 ,$$

and $\mathbf{b}_i$ is a $(d_i + 1) \times 1$ matrix with its m-th element being

$$\sum_{t=1}^{T-1} \sum_{S=1}^{N-2} \left\{ \left[\nabla_t D_i[t,S] - \alpha_{0i} \left(\frac{\Delta_T}{\Delta_N^2} \right) \nabla_S^2 D_i[t,S] \right] S^{2m-3.2} \Delta_N^{2m} \right\}^2 .$$

Therefore, under AIC (see [1]), the set of optimal estimators $\hat{\alpha}_i, \hat{d}_i, \hat{p}_{i0}, \ldots, \hat{p}_{id_i}$ minimizes

$$\mathrm{AIC}_i := 2d_i - 2\ell_i(\alpha_i, p_{i0}, \ldots, p_{id_i}). \tag{5.4}$$

The algorithm to minimize AIC_i:

(1) $m := -\infty, \hat{d}_i := 0$;

(2) For $d_i \in \{1, \ldots, 10\}$:

   (a) Numerically solve the first order condition of (5.3) for optimal $\hat{\alpha}_i, \hat{p}_{i0}, \ldots, \hat{p}_{id_i}$;

   (b) Calculate $a :=$AIC(d_i) based on (5.4);

   (c) If $m > a$ then $m := a, \hat{d}_i := d_i$;

(3) The final $\hat{d}_i$ is optimal within $\{1, \ldots, 10\}$ with AIC m.

5.2.2 AIC/Least-MSE Estimation of u_{0A}, u_{0B}, ρ

We continue to assume $i = A, B$. Similarly we need further assumptions about the initial condition u_{0i}. Assume

$$u_{0i}(x) = xq_i(x)\exp(-\gamma_i x)$$

where q_i is a polynomial and $\gamma_i > 0$ is a parameter, so that u_{0i} has exponential decay at infinity. Then $c_i := \deg q_i$ and its coefficients $q_{i0}, \ldots, q_{ic_i}$ are also a model parameter that needs to be determined under AIC. Now, if we take the expectation conditional on S^* on both sides of the evolution equation of V_i, we get

$$\mathbb{E}[V_i(t,S)|S^*] = \int_0^\infty p_-(t,S,y)u_{0i}(y)dy + \int_0^t S^*(s)\int_0^\infty q(t-s,S,y)\mathbb{E}[V_i(s,y)|S^*]dyds.$$

This implies that $\mathbb{E}[V_i|S^*]$ satisfies a deterministic Stefan-type PDE, which means that although we are generally unable to find an explicit expression of the solution, we can find a sufficiently accurate numerical solution. For a given dataset D_A, D_B, P, the goal in this part is thus to develop an algorithm to minimize the mean-square errors against $\mathbb{E}[V_i|S^*]$ and the mid-price function (moving boundary) determined by it under AIC.

Given a set of parameters $c_A, c_B, q_{A0}, \ldots, q_{Ac_A}, q_{B0}, \ldots, q_{Bc_B}, \gamma_A, \gamma_B, \rho$, denote the solution to the deterministic Stefan-type PDE (i.e., when $W \equiv 0$) by $\bar{V}_A$ and $\bar{V}_B$, which can be obtained numerically. Then the variances (errors) come from two parts: those from the evolution of the limit order book, and those from the evolution of the mid-price. Let θ_0 be a predefined constant which is the weight of importance of how well mid-price fits against how well the limit order book fits. Namely, we shall minimize the weighted MSE

$$\mathrm{MSE} := \mathrm{MSE}_1 + \theta_0 \mathrm{MSE}_2$$

where

$$\mathrm{MSE}_1 := \frac{1}{2TN} \sum_{i \in \{A,B\}} \sum_{t=1}^{T} \sum_{S=1}^{N} \left[D_i(t,S) - \bar{V}_i(\Delta_T t, \Delta_N S) \right]^2,$$

and

$$\mathrm{MSE}_2 := \frac{1}{T} \sum_{t=1}^{T} \left[\rho P(t) - \frac{\bar{V}_A(\Delta_T t, S) + \bar{V}_B(\Delta_T t, S)}{\Delta_T} \right]^2.$$

Finally, similar to the AIC approach in the previous part, the goal is to minimize

$$c_A + c_B + \mathrm{MSE},$$

which can be done by traversing all possible values of the model parameters, numerically finding $\bar{V}_A$ and $\bar{V}_B$, computing the corresponding MSE's, and finally finding the recorded optimal parameters. Note that unlike in previous part, because of the nonlinearity of $\bar{V}_A$ and $\bar{V}_B$, it is difficult to find a direct way to (numerically) solve for optimal coefficients for q_i, as opposed to p_i.

5.3 Optimization

Using the model combined with a set of optimal parameters adjusted for a given dataset, an investor can optimize his or her allocation of the amount of investment in the asset against consumption within a given amount of wealth. To analyze the investment behavior under our model, we assume the investor can only buy a single asset by making limit order transactions, that is, fulfilling ask orders placed by other sellers. This scenario is typical when an investor is avert to the volatility of price changes in market order transactions, or there lacks sufficient liquidity of the asset but the investor still has a strong motivation to make purchases.

Let $U : \mathbb{R}_+^3 \to \mathbb{R}$ be the utility function of the investor, which is monotone increasing and concave down on the second and third parameters. Specifically, $U(t, L_t, C_t)$ denotes the amount of happiness the investor obtains at time t if having the amount of asset L_t and consumption C_t. Since the investor would choose to pay the lowest possible price to buy an amount of asset, the amount of asset L_t and the total cost to buy such amount is completely determined by the highest limit price $S^*(t) + B$ the investor is willing to pay, which is illustrated in Figure 5.2.

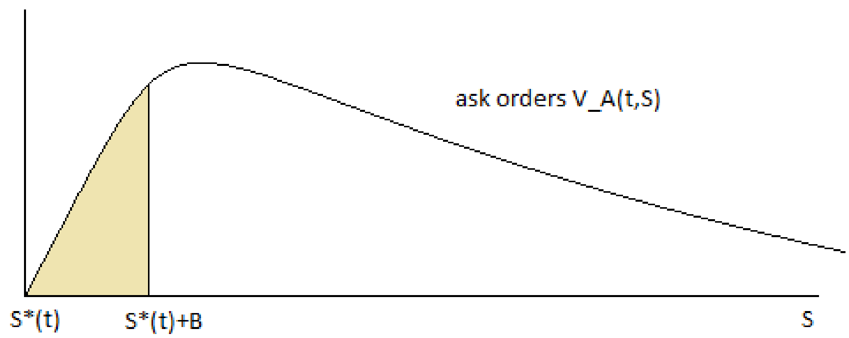

Figure 5.2: Investment Optimization via Limit Orders

Taking the model (and its parameters) as exogenous, and assuming the time t wealth is W_t, we then have the following optimization problem at time t (where the subscription t in the expectation denotes it is conditional to all information up to time t):

$$\max_{B\geq 0} \mathbb{E}_t \left[U\left(t, L_t(B), C_t\right) \right]$$

where

$$L_t(B) = \int_{S^*(t)}^{S^*(t)+B} V_A(t,S) dS$$

subject to the budget constraint

$$W_t = C_t + \int_{S^*(t)}^{S^*(t)+B} S V_A(t,S) dS.$$

When t is fixed, the optimization of U with respect to the choice of B is done through a static analysis; if we consider the full model, that is, both t and B vary, we come up with a dynamic analysis.

5.3.1 Static Analysis

Let time t be fixed. Denote the partial derivatives of U by U_t, U_L, U_C. Then we compute

$$\begin{aligned}\frac{\partial U}{\partial B}(t, L_t(B), C_t) = U_L(t, L_t(B), C_t) V_A(t, S^*(t) + B) \\ - U_C(t, L_t(B), C_t)(S^*(t) + B) V_A(t, S^*(t) + B).\end{aligned}$$

Therefore the optimal $B = B^*$ satisfies the first order condition

$$S^*(t) + B^* = \frac{U_L(t, L_t(B^*), C_t)}{U_C(t, L_t(B^*), C_t)}.$$

This means

Theorem 5.3.1 *In static optimization where the time t is fixed, the optimal highest limit price to buy the asset is equal to the ratio of the marginal utility of amount of asset to that of consumption.*

Proof As above. □

Note that $S^*(t) + B^*$ can be seen as the highest amount of money the investor is willing to pay to substitute 1 unit of asset with the same amount of consumption.

5.3.2 Dynamic Analysis

Now consider that t also varies, and we are interested in $\mathbb{E}_t[dU]$ from t to $t + dt$. Indeed, if we have $\mathbb{E}_t[dU(t, L_t(B^*), C_t)] > 0$, then even if $B = B^*$ reaches its static optimality, the investor would still wait for an amount of time to maximize the utility.

First we consider the evolution of $L_t(B)$. Fixing B and plugging in dV_A, we have formally

$$\begin{aligned} dL_t(B) = {} & \alpha_A \left[\frac{\partial V_A}{\partial S}(t, S^*(t) + B) - \frac{\partial V_A}{\partial S}(t, S^*(t)) \right] dt \\ & + \int_{S^*(t)}^{S^*(t)+B} \sigma(S - S^*(t)) W(dSdt) \\ & + \left[\int_0^t \int_{S^*(t)}^{S^*(t)+B} \sigma(S - S^*(t)) W(dSdt) \right] \dot{S}^*(t) dt. \end{aligned}$$

Similarly,

$$\begin{aligned} dC_t = {} & \alpha_A \left[V_A(t, S^*(t) + B) - (S^*(t) + B) \frac{\partial V_A}{\partial S}(t, S^*(t) + B) + S^*(t) \frac{\partial V_A}{\partial S}(t, S^*(t)) \right] dt \\ & + \int_{S^*(t)}^{S^*(t)+B} S\sigma(S - S^*(t)) W(dSdt) \\ & + \left[\int_0^t \int_{S^*(t)}^{S^*(t)+B} S\sigma(S - S^*(t)) W(dSdt) \right] \dot{S}^*(t) dt. \end{aligned}$$

Then by Itô's formula, and noting that $S^*(t)$ is part of the information at t, we get

$$\begin{aligned}\mathbb{E}_t[dU] &= U_t dt + U_L \alpha_A \left[\frac{\partial V_A}{\partial S}(t, S^*(t)+B) - \frac{\partial V_A}{\partial S}(t, S^*(t))\right] dt + \frac{1}{2} U_{LL} \int_0^B \sigma^2(S) dS dt \\ &+ U_C \alpha_A \left[V_A(t, S^*(t)+B) - (S^*(t)+B)\frac{\partial V_A}{\partial S}(t, S^*(t)+B) + S^*(t)\frac{\partial V_A}{\partial S}(t, S^*(t))\right] dt \\ &+ \frac{1}{2} U_{CC} \int_0^B (S^*(t)+S)^2 \sigma^2(S) dS dt.\end{aligned}$$

Since for a fixed t we are only interested at B^* in Theorem 5.3.1, and $S^* + B^* = U_L/U_C$, we have

$$\begin{aligned}\mathbb{E}_t\left[\frac{dU}{dt}(t, L_t(B^*), C_t)\right] &= U_t + U_C \alpha_A \left[V_A(t, S^*(t)+B^*) - B^* \frac{\partial V_A}{\partial S}(t, S^*(t))\right] \\ &+ \frac{1}{2} U_{LL} \int_0^{B^*} \sigma^2(S) dS + \frac{1}{2} U_{CC} \int_0^{B^*} (S^*(t)+S)^2 \sigma^2(S) dS.\end{aligned} \tag{5.5}$$

If we denote the absolute risk aversions of the investor with respect to the amount of asset and the consumption by

$$\begin{aligned} r_L &:= -\frac{U_{LL}}{U_L}, \\ r_C &:= -\frac{U_{CC}}{U_C}, \end{aligned}$$

then (5.5) is equivalent to

$$\begin{aligned}\mathbb{E}_t\left[\frac{dU}{dt}(t, L_t(B^*), C_t)\right] = U_t + U_C \Bigg\{ &\alpha_A \left[V_A(t, S^*(t)+B^*) - B^* \frac{\partial V_A}{\partial S}(t, S^*(t))\right] \\ &- \frac{1}{2} r_L (S^*+B^*) \int_0^{B^*} \sigma^2(S) dS - \frac{1}{2} r_C \int_0^{B^*} (S^*+S)^2 \sigma^2(S) dS \Bigg\}.\end{aligned} \tag{5.6}$$

The investor would then use the dataset D_A to compute $\mathbb{E}_t\left[\frac{dU}{dt}\right]$ and see the expected change of U from t to $t+dt$ at B^*. Specifically, we have the following observations from Equation (5.5) or (5.6):

(1) the larger the quantity

$$V_A(t, S^*(t)+B^*) - B^* \frac{\partial V_A}{\partial S}(t, S^*(t))$$

is, the more possible it is for U to increase;

(2) the greater absolute risk aversions r_L and r_C the investor has, the more possible it is for U to decrease;

(3) the larger B^* is, the more risk U is exposed to for a decrease, since both $\int_0^{B^*} \sigma^2(S) dS$ and $\int_0^{B^*} (S^* + S)^2 \sigma^2(S) dS$ are increasing functions of B^*.

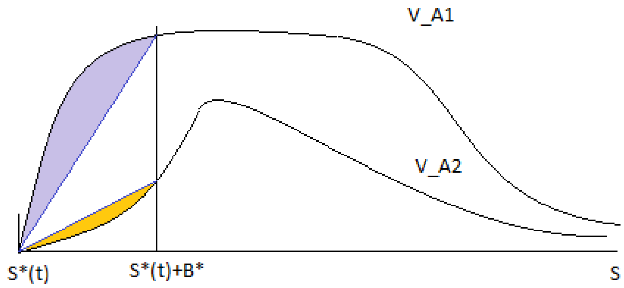

Figure 5.3: An Illustration of Theorem 5.3.2

Also, if we make two natural assumptions about U,

(a) the utility function is discounted in time, that is, $U_t < 0$,

(b) the investor is risk avert or risk neutral, that is, $U_{LL} \leq 0, U_{CC} \leq 0$,

then we have

Theorem 5.3.2 *Suppose B^* is the statically optimal choice at time t as in Theorem 5.3.1. If*

$$V_A(t, S^*(t) + B^*) \leq B^* \frac{\partial V_A}{\partial S}(t, S^*(t)), \tag{5.7}$$

then

$$\mathbb{E}_t \left[\frac{dU}{dt}(t, L_t(B^*), C_t) \right] \leq 0.$$

Proof As above. □

Note that (5.7) in Theorem 5.3.2 has an intuitive illustration in Figure 5.3. The quantity $V_A(t, S^*(t) + B^*)/B^*$ is the slope of the colored lines between the mid-price point $(S^*(t), 0)$ and the optimal point $(S^* + B^*, V_A(t, S^* + B^*))$.

* If it is less than the boundary derivative $\partial V_A(t, S^*(t))/\partial S$ as shown in curve V_{A1}, then the utility at next instant $t + dt$ is expected to drop, as the volume of ask orders at lower limit prices between S^* and $S^* + B^*$ tends to fall, so it is best to make the purchase at t rather than wait.

* On the other hand, as shown in curve V_{A2}, the volume at lower prices tends to rise, and it might be wise to wait for a future time to make the purchase. In this case the investor needs to evaluate other factors such as time discount U_t and risk premium terms as well.

Chapter 6

Summary

In this chapter we summarize the main results obtained in the previous chapters. Throughout this chapter we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose $W : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a standard 2-dimensional Brownian sheet.

6.1 Mathematical Results

In the mathematics of this thesis we studied 3 types of stochastic equations.

6.1.1 A Stochastic Stefan Problem with Spatially Colored Noise

The solution $u(t,x)$ to the stochastic Stefan equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + u(t,x)d\xi_t(x), \forall x > \beta(t),\\
u(t,x) &= 0, \forall x \leq \beta(t),\\
u(0,x) &= u_0(x),\\
\rho\dot{\beta}(t) &= \lim_{x \searrow \beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{6.1}$$

exists and is unique for $0 \leq t < \tau := \lim_{L\to\infty} \tau^L$ where $\tau^L := \inf\{t \in \mathbb{R}_+ \cup \{0\} : |\dot{\beta}(t)| \geq L\}$.

6.1.2 Boundary Regularity of the Stochastic Heat Equation

The solution $u(t,x)$ to the stochastic heat equation

$$\begin{aligned}
\frac{\partial u}{\partial x}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial^2 W}{\partial t \partial x}, \forall x > 0, t \in [0,T],\\
u(0,x) &= u_0(x),\\
u(t,0) &= 0, \forall x \leq 0, t \in [0,T]
\end{aligned} \tag{6.2}$$

exists and is unique; Moreover, define $v(t,x) := u(t,x)/x$, then $\mathbb{P}$-a.s.,

$$\lim_{x\searrow 0} v(t,x) = \frac{\partial u}{\partial x}(t,0+) = \int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u_0(y)dy + \int_0^t \int_0^\infty \frac{\partial p}{\partial x}(t,0,y)u(s,y)W(dyds);$$

If we further define $v(t,0) := \lim_{x\searrow 0} v(t,x)$, then $\mathbb{P}$-a.s., $v(t,x)$ is $\left(\frac{1}{4}-\epsilon, \frac{1}{6}-\epsilon\right)$-Hölder continuous on $[0,T]\times[0,1]$ for $\epsilon > 0$.

6.1.3 A Stochastic Stefan Problem with Space-Time Brownian Noise

Let $\sigma : [0,\infty) \to \mathbb{R}$ be a regular scaling function (see the definition given in Chapter 4). Then the solution $u(t,x)$ to the stochastic Stefan equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} + \sigma(x-\beta(t))\frac{\partial^2 W}{\partial t\partial x}, \forall x > \beta(t),\\
u(t,x) &= 0, \forall x \le \beta(t),\\
u(0,x) &= u_0(x),\\
\rho\dot{\beta}(t) &= \lim_{x\searrow\beta(t)} \frac{u(t,x)}{x}, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{6.3}$$

exists and is unique for $0 \le t < \tau$ for $0 \le t < \tau := \lim_{L\to\infty} \tau^L$ where $\tau^L := \inf\{t \in \mathbb{R}_+ \cup \{0\} : |\dot{\beta}(t)| \ge L\}$; Moreover, $\mathbb{P}$-a.s., $\beta \in C^1([0,\tau))$ and $\dot{\beta}(t)$ is $\left(\frac{1}{4}-\epsilon\right)$-Hölder continuous for $\epsilon > 0$.

6.2 Modeling of Market Limit Orders

For a set of given model parameters $u_{0A}, u_{0B}, \alpha_A, \alpha_B, \sigma_A, \sigma_B, \rho$, the model

$$\begin{aligned}
\frac{\partial V_A}{\partial t} &= \alpha_A \frac{\partial^2 V_A}{\partial S^2} + \sigma_A(|S - S^*(t)|)\frac{\partial^2 W}{\partial t\partial S}, \forall S > S^*(t),\\
V_A(t,S) &= 0, \forall S \le S^*(t),\\
V_A(0,S) &= u_{0A}(S),\\
\frac{\partial V_B}{\partial t} &= \alpha_B \frac{\partial^2 V_B}{\partial S^2} + \sigma_B(|S - S^*(t)|)\frac{\partial^2 W}{\partial t\partial S}, \forall S < S^*(t),\\
V_B(t,S) &= 0, \forall S \ge S^*(t),\\
V_B(0,S) &= u_{0B}(S),\\
\rho\frac{dS^*}{dt}(t) &= \left[\frac{\partial V_A}{\partial S}(t,S^*(t)+) + \frac{\partial V_B}{\partial S}(t,S^*(t)-)\right]
\end{aligned} \tag{6.4}$$

exists and is unique.

6.2.1 Parameter Estimation

Suppose P, D_A, D_B is a given limit order dataset. For $i = A, B$, suppose

$$\sigma_i(x) := \frac{x^{1.6}}{1 + xp_i(x)}, i = A, B$$

where $p_i = \sum_{j=0}^{d_i} p_{ij} x^j$ is a polynomial with $\deg p_i \geq 1$. Then the first step is to minimize for $i = A, B$,

$$\begin{aligned} \mathrm{AIC}_i &:= 2d_i - 2\ell_i(\alpha_i, p_{i0}, \ldots, p_{id_i}) \\ &= \sum_{t=1}^{T-1} \sum_{S=1}^{N-2} \left\{ \left[\nabla_t D_i[t, S] - \alpha_i \left(\frac{\Delta_T}{\Delta_N^2} \right) \nabla_S^2 D_i[t, S] \right] \frac{1 + \sum_{j=0}^{d_i} p_{ij} S^{j+1} \Delta_N^{j+1}}{S^{1.6} \Delta_N^{1.6}} \right\}^2 . \end{aligned} \tag{6.5}$$

Suppose also

$$u_{0i}(x) = xq_i(x) \exp(-\gamma_i x)$$

where $q_i = \sum_{j=0}^{c_i} q_{ij} x^j$ is a polynomial and $\gamma_i > 0$ is a parameter. Then the second step is for a predefined weight θ_0 to minimize

$$c_A + c_B + \mathrm{MSE}_1 + \theta_0 \mathrm{MSE}_2$$

where

$$\begin{aligned} \mathrm{MSE}_1 &:= \frac{1}{2TN} \sum_{i \in \{A,B\}} \sum_{t=1}^{T} \sum_{S=1}^{N} \left[D_i(t, S) - \bar{V}_i(\Delta_T t, \Delta_N S) \right]^2 , \\ \mathrm{MSE}_2 &:= \frac{1}{T} \sum_{t=1}^{T} \left[\rho P(t) - \frac{\bar{V}_A(\Delta_T t, S) + \bar{V}_B(\Delta_T t, S)}{\Delta_T} \right]^2 , \end{aligned}$$

and $\bar{V}_i(t, x)$ is the numerical solution by setting $W \equiv 0$.

6.2.2 Optimization

Suppose $B^*(t)$ is the statically optimal price at time t with respect to the utility function U and wealth W_t. Then

$$S^*(t) + B^*(t) = \frac{U_L(t, L_t(B^*(t)), C_t)}{U_C(t, L_t(B^*(t)), C_t)}.$$

In other words, the optimal highest limit price to buy the asset is equal to the ratio of the marginal utility of amount of asset to that of consumption.

Also, let r_L, r_C be the absolute risk aversions with respect to L, C, then we have

$$\begin{aligned}\mathbb{E}_t\left[\frac{dU}{dt}(t, L_t(B^*), C_t)\right] = U_t + U_C\Bigg\{\alpha_A\left[V_A(t, S^*(t)+B^*) - B^*\frac{\partial V_A}{\partial S}(t, S^*(t))\right] \\ -\frac{1}{2}r_L(S^*+B^*)\int_0^{B^*}\sigma^2(S)dS - \frac{1}{2}r_C\int_0^{B^*}(S^*+S)^2\sigma^2(S)dS\Bigg\}.\end{aligned} \tag{6.6}$$

In particular, if

$$\frac{V_A(t, S^*(t)+B^*)}{B^*} \leq \frac{\partial V_A}{\partial S}(t, S^*(t)), \tag{6.7}$$

then

$$\mathbb{E}_t\left[\frac{dU}{dt}(t, L_t(B^*), C_t)\right] \leq 0.$$

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